2605

John Rutland Esq

Ex Lib: R: H:

2605

John Rutland Esq

Ex Lib: R: H:

L

8530.aaa.28.

Keill (J)

THE ELEMENTS. Of PLAIN and SPHERICAL TRIGONOMETRY.

Alſo a Short

TREATISE OF THE NATURE and ARITHMETICK OF LOGARITHMS.

By Doctor **John Keil**, F. R. S. And late Profeſſor of **Aſtronomy** in **Oxford**.

The Third Edition, Tranſlated by **Samuel Cunn**.
And carefully Corrected by **S. Fuller**.

DUBLIN:
Printed by W. Wilmot on the Blind-Key, for
Samuel Fuller, at the Globe in Meath-Street,
and Sold by the Book-Sellers, MDCCXXVI.

BRITISH MUSEUM

TO THE

LEARNED,

THE

Provoſt, Fellows,

AND

SCHOLARS

OF THE

COLLEGE of DUBLIN.

This Edition

Of Dr. KEILS Trigonometry

IS HUMBLY

DEDICATED

BY THE

EDITOR.

MATHEMATICAL Books, Sold by *SAMUEL FULLER*, at the *GLOBE* in *MEATH-STREET*. 1726.

ARITHMETICK

by

ALingham
Ayres
Baker
Cocker
Collins
Cunn
Foſter
Hallyman
Hatton
Hewit
Hill
Johnſon
Jordan
Leybourn
Lydal
Author of Mercantile, Arith.
Moor
Moreland
Newton
Parſons
Paſcal Record
Richardſon
Shelley
Speidal
Tacquet
Wells
Wilſon
Wingate.

MERCHANTS ACC.

by

COllins
Dafforne
Gentleman-Account.
Hatton
Hawkins
King
Author of Lex Mercatoria.
Malcolm
Nevel
Scarlet
Snell
Vernon
Wilſon.

CHRONOLOGY

by

BUcholcherus
Niſbet
Petavius
Strauchius
Swan
Uſher
Wells

GEOMETRY, OR MENSURATION

by

ALingham
Alphonſus
Anderſon
Barrow
Bonnel
Cogſhal
Cohorn
Dechales
Euclid
Hawney
Leybourn
Moor
Pardie
Record
Rudd
Tacquet
Vauban
Wallis
Wells
Whiſton

GAGING

by

ANderſon
Brown
Clayton
Collins
Cumpſty
Davis
Everard

GAGING

Eyers
Hunt
Jones
Light-body
Smith
Wibrand.

TRIGONOMETRY.

by

Foster
Gellibrand
Harris
Hawney
Heins
Keil
Napier
Newhouſe
Newton
Norwood
Pitiſcus
Sherwin.

NAVIGATION

by

Atkinſon
Blackborow
Bond
Collins
Colſon
Gellibrand
Hardingham
Hawney

NAVIGATION

Hodgſon
Jones
Kelly
Newhouſe
Newton
Norwood
Perkins
Phillips
Seller
Sturmy
Wilſon
Wright

ASTRONOMY

or

ASTROLOGY

by

Alphonſus
Argol
Baker
Ball
Biſhop
Blagrave. Blaeu.
Blundevil
Cattan
Colon
Culpepper
Edlin
Gadbury
Galileo
Gaſſendus
Gemma Friſius

ASTRONOMY

Goad
Goclenius
Gregory
Greenwood
Halley
Haly. Hanna.
Harris
Hues
Huigens
Keil
Kepler
Lamb
Lansbergius
Leybourn
Lilly
Mæſtlin
Manilius
Mercator
Middleton
Morden
Morinus
Moxon
Newton
Partridge
Proclus
Ptolomy
Ramſay
Saunders
Spark
Street
Ubaldus
Wells
Wharton
Whiſton
Wing
Wright

GEO-

GEOGRAPHY

by

Blome
Cambden
Carpenter
Clark
Cluverius
Echard
Fer
Gordon
Gregory
Mead
Meriton
Moll
Morden
Newton
Senex
Speed
Varenius
Wells

SURVEYING

by

Burgh
Holwel
Hopton
Kircher
Lawrence
Leybourn
Love
Martindale
Rathborn
Shaw
Wing

ALGEBRA

by

Lexander
Baker
Branker
Cotes
Ditton
Gibſon
Harris
Jones
Kerſey
Newton
Oughtred
Parſons
Record
Ronayne
Sault
Sturmy
Wallis
Ward
Wells

DIALLING

by

Collins
Dela-Hire
Foſter
Good
King
Leybourn
Marius
Serles
Stirrup
Wells
Wilſon

ARCHITECTURE

by

Brown
Evelin
Halfpenny
Le-Clerk
Leybourn
Neve
Palladio
Scamozzi

MECHANICKS

by

Derham
Mandy
Mariot
Moxon
Rohault
Wats

PERSP.

or

OPTICKS &c.

by

Ditton
Graveſand
Gregory
Moxon
Newton
Taylor

FLUXIONS or CONIC's by

Angelis
Apollonius
Collins
Dela-Hire
Ditton
Hayes
Hoſpital
Mc. Laurin
Milnes
Newton
Robinſon
Schooten
Steel
Taylor
Theodoſius
Greg. Vincent.

Miſcellaneous Tracts, or a *Courſe* of the *MATHEMATICKS.*

BY

Blundevil
Cheyne
De-Chales
Derham
Foſter
Galtruchius
Graveſand
Gunter
Harris
Hawksbee
Hawney
Herigonius
Johnſton
Jones
Keil
La-Motte
Leybourn
Louthorp
Lucar
Mandy
Maralois
Moor
Author of Miſcellanea Curioſa.
Neve
Newton
Ozanam
Perkins
Pickering
Rohault
The Royal Society
Taſſius
Taylor
Ward
Wells
Wilkins

☞ *Note*, Books Bought and Sold, by the aforeſaid *S. Fuller*, at reaſonable Rates. Alſo ready Money for old Libraries.

THE ELEMENTS Of *Plain* and *Spherical* TRIGONOMETRY.

DEFINITIONS.

THE Buſineſs of Trigonometry is to find the Angles, when the Sides are given, and the Sides, or the Ratio's of the Sides, when the Angles are given, and to find Sides and Angles, when Sides and Angles are given: In order to which, it is neceſſary that not only the Peripheries of Circles, but alſo certain Right Lines in and about Circles be ſuppoſed divided into ſome determined Number of Parts.

And ſo the Ancient Mathematicians thought fit to divide the Periphery of a Circle into 360 *Parts (which they call Degrees;) and every*

Degree into 60 Minutes, and every Minute into 60 Seconds: And again, every Second into 60 Thirds, and ſo on. And every Angle is ſaid to be of ſuch a Number of Degrees and Minutes, as there are in the Arc meaſuring that Angle.

There are ſome that would have a Degree divided into centeſimal Parts, rather than ſexageſimal ones: And it would perhaps be more uſeful to divide, not only a Degree, but even the whole Circle in a decuple Ratio; which Diviſion may ſome time or other gain Place. Now, if a Circle contain 360 Degrees, a Quadrant thereof, which is the Meaſure of a Right Angle, will be 90 of thoſe Parts: And if it contains 100 Parts, a Quadrant will be 25 of theſe Parts.

The Complement *of an Arc is the Difference thereof from a Quadrant.*

A Chord, *or* Subtenſe, *is a Right Line drawn from one End of the Arc to the other.*

The Right Sine *of any Arc, which alſo is commonly called only a Sine, is a Perpendicular falling from one End of an Arc, to the Radius down thro' the other End of the ſaid Arc. And is therefore the Semiſubtenſe of double the Arc,* viz. $DE = \frac{1}{2} DO$, *and the Arc* DO *is double of the Arc* DB. *Hence, the Sine of an Arc of* 30 *Degrees, is equal to the one half of the Radius. For* (by 15. *El.* 4.) *the Side of an Hexagon inſcribed in a Circle, that is, the Subtenſe of* 60 *Degrees is equal to the Radius. A Sine divides the Radius into two Segments* CE, EB; *one of which,* CE, *which is intercepted between the Center and the Right Sine, is the Sine Complement of the Arc* DB *to a Quadrant, (for*

CE

C E=F D which is the Sine of the Arc D H,) and is called the Coſine. *The other Segment B E, which is intercepted between the Right Sine and the Periphery, is called a* Verſed Sine, *and ſometimes a* Sagitta.

And if the Right Line C G be produced from the Center C, thro' one End D of the Arc, until it meets the Right Line B G, which is perpendicular to the Diameter drawn thro' the other End B of the Arc, then C G is called the Secant, *and B G the* Tangent *of the Arc D B.*

The Coſecant and Cotangent of an Arc is the Secant or Tangent of that Arc, which is the Complement of the former Arc to a Quadrant. Note, *As the Chord of an Arc, and of it's Complement to a Circle, is the ſame; ſo likewiſe is the Sine, Tangent, and Secant of an Arc the ſame as the Sine, Tangent, and Secant of the Complement to a Semicircle.*

The Sinus Totus *is the greateſt Sine, or the Sine of 90 Degrees, which is equal to the Radius of the Circle.*

A Trigonometrical Canon *is a Table, which, beginning from one Minute, orderly expreſſes the Lengths that every Sine, Tangent, and Secant, have in reſpect of the Radius, which is ſuppoſed Unity; and is conceived to be divided into 10,000,000, or more decimal Parts. And ſo the Sine, Tangent, or Secant of any Arc, may be had by help of this Table; and contrariwiſe, a Sine, Tangent, or Secant, being given, we may find the Arc it expreſſes.* Take Notice, *That in the following Tract,* R *ſignifies the Radius,* S *a Sine,* Coſ. *a Coſine,* T *a Tangent, and* Cot. *a Cotangent.*

The CONSTRUCTION of the *Trigonometrical Canon.*

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is alſo given.

FOR (by 47 of the firſt *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$. and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the ſquare Root, there is given $AC = \sqrt{ABq + BCq}$ and $AB = \sqrt{ACq - BCq}$. And $BC = \sqrt{ACq - ABq}$.

PROPOSITION II.

PROBLEM.

The Sine DE of the Arc DB, and the Radius CD, being given, to find the Coſine DF.

THE Radius CD and the Sine DE, being given in the Right-angled Triangle CDE, there will be given (by the laſt *Prop.*) $CE = \sqrt{CDq - DEq} = DF$.

PROPO-

PROPOSITION III.

PROBLEM.

The Sine D E of any Arc D B being given, to find D M or B M the Sine of half the Arc.

D E being given, C E (by the laſt *Prop.*) will be given, and accordingly E B which is the Difference between the Coſine and Radius. Therefore D E, E B, being given in the Right-angled Triangle D B E, there will be given D B, whoſe half D M is the Sine of the Arc D L $=\frac{1}{2}$ the Arc B D.

PROPOSITION IV.

PROBLEM.

The Sine B M of the Arc B L being given, to find the Sine of double that Arc.

THE Sine B M being given, there will be given (by *Prop.* 2.) the Coſine C M. But the Triangles C B M, D B E, are equiangular, becauſe the Angles at E and M are Right Angles, and the Angle at B common. Wherefore (by 4. *6.*) we have C B : C M :: B D, or 2 B M : D E. Whence, ſince the three firſt Terms of this Analogy are given, the fourth alſo, which is the Sine of the Arc D B, will be known.

Coroll. Hence, C B : 2 C M :: B D : 2 D E, that is, the Radius is to double the Coſine of one half of the Arc D B, as the Subtenſe of the Arc D B, is to the Subtenſe of double that Arc. Alſo C B : 2 C M : : (2 B M : 2 D E : :) B M : D E : : $\frac{1}{2}$ C B : C M. Wherefore the Sine of any Arc, and the Sine of it's Double being given, the Coſine of the Arc it ſelf is given.

PROPOSITION V.

The Sines of two Arcs B D, F D, being given, to find F I the Sine of the Sum, as likewiſe E L, the Sine of their Difference.

LET the Radius C D be drawn, and then C O is the Coſine of the Arc F D, which accordingly is given, and draw O P thro' O parallel to D K. Alſo let O M, G E, be drawn parallel to C B. Then becauſe the Triangles C D K, C O P, C H I, F O H, F O M, are equiangular. In the firſt Place, C D : D K : : C O : O P, which Conſequently is known. Alſo we have C D : C K : : F O : F M, and ſo likewiſe this ſhall be known. But becauſe F O = E O, then will F M = M G = O N. And ſo O P + F M = F I = Sine of the Sum of the Arcs : And O P — F M, that is, O P — O N = E L = Sine of the Difference of the Arcs. W. W. D.

Coroll. Becauſe the Differences of the Arcs B E, B D, B F, are equal, the Arc B D ſhall be an Arithmetical Mean between the Arcs B E, B F.

PROPO-

PROPOSITION VI.

The ſame Things being ſuppoſed, Radius is to double the Coſine of the mean Arc, as the Sine of the Difference, to the Difference of the Sines of the Extremes.

FOR we have CD : CK :: FO : FM, whence by doubling the Conſequents CD : 2CK :: FO : 2FM, or to FG; which is the Difference of the Sines EL, FI. W. W. D.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI, EL, ſhall be equal to FO the Sine of the Diſtance. For in this Caſe, CK, is the Sine of 30 Degrees, Double thereof being equal to Radius; and ſo ſince CD = 2CK, we ſhall have FO = FG. And conſequently, if the two Arcs BE, BF, are Equidiſtant from the Arc of 60 Degrees, the Difference of the Sines ſhall be equal to the Sine of the Diſtance FD.

Coroll. 2. Hence, if the Sines of all Arcs be given, diſtant from one another by a given Interval, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees = Sine of 59 Degrees + Sine of 1 Degree. And the Sine of 62 Degrees = Sine of 58 Degrees + Sine of 2 Degrees. Alſo the Sine of 63 Degrees = Sine of 57 Degrees + Sine of 3 Degrees, and ſo on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, diſtant from each other, by a given Interval be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the preceding Analogy, all the Sines to 30 Degrees, may be found. For Radius is to double the Coſine of 15 Degrees, as the Sine of 1 Degree, is to the Difference of the Sines of 14 Degrees, and 16 Degrees; ſo alſo is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and ſo on continually until you come to the Sine of 30 Degrees.

After the ſame Manner, as Radius is to double the Coſine of 30 Degrees, or to double the Sine of 60 Degrees, ſo is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees : : Sine 2 Degrees, to the Difference of the Sines of 28 and 32 Degrees : : Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But in this Caſe, Radius is to double the Coſine of 30 Degrees, as 1 to $\sqrt{3}$. And accordingly, if the Sines of the Diſtances from the Arc of 30 Degrees, be multiply'd by $\sqrt{3}$, the Differences of the Sines will be had.

So likewiſe may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Coſines of one and two Minutes given. For as the Radius is to double the Coſine of 2′ : : Sine 1′ : Difference of the Sines of 1′ and 3′ : : Sine 2′ : Difference of the Sines of 0′ and 4′, that is, to the Sine of 4′. And ſo the Sines of the four firſt Minutes being given; we may thereby find the Sines of the others to 8′, and from thence to 16′, and ſo on.

PROPOSITION VII.

THEOREM.

In ſmall Arcs, the Sines and Tangents of the ſame Arcs are nearly to one another, in a Ratio of Equality.

FOR becauſe the Triangles CED, CBG, are equiangular, CE : CB : : ED : BG. But as the Point E approaches B, EB will vaniſh in Reſpect of the Arc BD: Whence CE will become nearly equal to CB. And ſo ED will be alſo nearly equal to BG. If EB be leſs than the $\frac{1}{10{,}000{,}000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be alſo leſs than the $\frac{1}{10{,}000{,}000}$ Part of the Tangent.

Coroll. Since any Arc is leſs than the Tangent, and greater than it's Sine, and the Sine and Tangent of a very ſmall Arc, are nearly equal; it follows that the Arc ſhall be nearly ly equal to it's Sine; and ſo in very ſmall Arcs it ſhall be, as Arc is to Arc, ſo is Sine to Sine.

PROPOSITION VIII.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inſcribed in a Circle, that is, the Subtenſe of 60 Degrees, is equal to the Radius, (*by* 15*th* of the 4*th*,) and ſo the half of the Radius ſhall be the Sine of the Arc 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found, (*by Prop.* 3.) Alſo the Sine of the Arc of 15 Degrees being given, (*by the ſame Prop.*) we may have the Sine of 7 Degrees 30 Minutes: So likewiſe can we find the Sine of the half of this, *viz.* 3 Degrees 45′; and ſo on, until twelve Biſections being made, we come to an Arc of 52^{2}, 44^{3}, 3^{4}, 45^{5}, whoſe Coſine is nearly equal to the Radius, in which Caſe (*as is manifeſt from Prop.* 7.) Arcs are proportional to their Sines: And ſo, as the Arc of 52^{2}, 44^{3}, 3^{4}, 45^{5}, is to an Arc of one Minute, ſo ſhall the Sine before found, be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop.* 2. *and* 4.) the Sine and Coſine of two Minutes will be had.

PROPO-

PROPOSITION IX.

THEOREM.

If the Angle BAC, *being in the Periphery of a Circle, be biſected by the Right Line* AD, *and if* AC *be produced until* $DE = AD$ *meets it in* E: *then ſhall* $CE = AB$.

IN the quadrilateral Figure ABDC (*by* 22. 3.) the Angles B and ACD are equal to two Right Angles $= DCE + DCA$ (*by* 13. 1.) Whence the Angle $B = DCE$. But likewiſe the Angle $E = DAC$ (*by* 5. 1.) $= DAB$ and $DC = DB$. Wherefore the Triangles BAD and CED are congruous, and CE is equal to AB W. W. D.

PROPOSITION X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, *&c. be equal; and let the Subtenſes of the Arcs* AB, AC, AD, AE, &c. *be drawn; then will* $AB : AC :: AC : AB + AD :: AD : AC + AE :: AE : AD + AF :: AF : AE + AG$.

LET AD be produced to H, AE to I, AF to K, and AG to L, that the Triangles ACH, ADI, AEK, AFL, be Iſoſceles ones; then becauſe the Angle BAD is biſected, we ſhall have D H = A B, (*by the laſt Prop.*) ſo likewiſe ſhall E I = A C, F K = A D, alſo G L = A E.

But the Iſoſceles Triangles A B C, C H A, D A I, E A K, F A L, becauſe of the equal Angles at the Baſes, are equiangular. Wherefore it ſhall be as $AB : AC :: AC : AH = AB + AD :: AD : AI = AC + AE :: AE : AK = AD + AF :: AF : AL = AE + AG$. W. W. D.

Coroll. 1. Becauſe A B is to A C, as Radius is to double the Coſine of $\frac{1}{2}$ the Arc A B, it ſhall alſo be (by *Corol. Prop.* 4.) as Radius is to double the Coſine of $\frac{1}{2}$ the Arc A B, ſo is $\frac{1}{2}AB : \frac{1}{2}AC :: \frac{1}{2}AC : \frac{1}{2}AB + \frac{1}{2}AD :: \frac{1}{2}AD : \frac{1}{2}AC + \frac{1}{2}AE :: \frac{1}{2}AE : \frac{1}{2}AD + \frac{1}{2}AF$, *&c.* Now let each of the Arcs A B, B C, C D, *&c.* be 2′ then will $\frac{1}{2}$A B be the Sine of one Minute, $\frac{1}{2}$A C the Sine of 2′ Minutes, $\frac{1}{2}$A D the Sine of 3′ Minutes; $\frac{1}{2}$ A E the Sine of 4′, *&c.* Whence if the Sines of one and two Minutes be given, we may eaſily find all the other Sines in the following Manner.

Let the Coſine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. 59′, be called Q, and make the following Analogies; R : 2 Q : : Sin. 2′ : S. 1′ + S. 3′. Wherefore the Sine of 3 Minutes will be given. Alſo R : 2 Q : : S. 3′ : S. 2′ + S. 4′. Wherefore the S. 4′ is given; and R : 2 Q : : S. 4′ : S. 3′ + S. 5′; and ſo the Sine of 5′ will be had.

Likewiſe R : 2 Q : : S. 5′ : S. 4′ + S. 6′ ; and ſo we ſhall have the Sine of 6′. And in the like Manner, the Sines of every Minute of the Quadrant will be given. And becauſe the Radius, or the firſt Term of the Analogy is Unity, the Operations will be with great Eaſe and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only, (by *Cor.* 1. *Prop.* 6.)

The Sines being given, the Tangents and Secants may be found from the following Analogies, (in the Figure for the Definitions ;) becauſe the Triangles C E D, C B G, C H I, are equiangular, we have.

C E : E D : : C B : B G ; that is, Coſ. : S : : R : T.
G B : B C : : C H : H I ; that is, T : R : : R : Cot.
C E : C D : : C B : C G ; that is, Coſ. : R : : R : Secant.
D E : C D : : C H : C I ; that is, S : R : : R : Coſec.

SCHOLIUM.

That great Geometrician and incomparable Philoſopher, Sir Iſaac Newton, *was the firſt that laid down a Series converging,* in infinitum ; *from which, having the Arcs given, their Sines may be found. Thus if an Arc be called* A, *and the Radius be an Unit, the Sine thereof will be found to be.*

$$A - \frac{A^3}{1.2.3.} + \frac{A^5}{1.2.3.4.5.} - \frac{A^7}{1.2.3.4.5.6.7.} + \frac{A^9}{1.2.3.4.5.6.7.8.9.} - \&c.$$

And the Coſine,

$$1 - \frac{A^2}{1.2.} + \frac{A^4}{1.2.3.4.} - \frac{A^6}{1.2.3.4.5.6.} + \frac{A^8}{1.2.3.4.5.6.7.8.} \text{ \&c.}$$

Theſe Series in the Beginning of the Quadrant when the Arc A *is but ſmall, ſoon converge. For in the Series for the Sine, if* A *does not exceed* 10 *Minutes, the two firſt Terms thereof,* viz. $A - \frac{1}{6}A^3$ *gives the Sine to* 15 *Places of Figures. If the Arc* A *be not greater than one Degree, the three firſt Terms will exhibit the Sine to* 15 *Places of Figures ; and ſo the ſaid Series are very uſeful for finding the firſt and laſt Sines of the Quadrant. But the greater the Arc* A *is, the more are the Terms of the Series required to have the Sine in Numbers true to a given Place of Figures. And then when the Arc is nearly Equal to the Radius, the Series Converges very ſlow. And therefore, to remedy this, I have deviſed other Series, ſimilar to the* Newtonian *one, wherein, I ſuppoſe, the Arc, whoſe Sine is ſought, is the Sum and Difference of two Arcs,* viz. $A + z$, *or* $A - z$: *And let the Sine of the Arc* A, *be called* a, *and the Coſine* b. *Then the Sine of the Arc* $A + z$ *will be expreſſed thus :*

$$1.\quad a + \frac{bz}{1.} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} \text{ \&c.}$$

And the Coſine is,

$$2.\quad b - \frac{az}{1.} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6} \text{ \&c.}$$

In like Manner the Sine of the Arc A — z *is*

$$3.\quad a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6}$$

&c.

And the Coſine is,

$$4.\quad b + \frac{az}{1} - \frac{bz^2}{1.2} - \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5} \quad \&c.$$

The Arc A *is an Arithmetical Mean between the* A−z and A+z. *And the Difference of the Sines are,*

$$5.\quad \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6} \quad \&c.$$

$$6.\quad \frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6}$$

&c.

Whence the Difference of the Differences or ſecond Difference.

$$7.\quad \textit{Produce}\ \frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6} \quad \&c.$$

$$\textit{Or,}\ 2\,a \times \frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} \quad \&c.$$

Which Series is equal to double the Sine of the Mean Arc, drawn into the verſed Sine of the Arc z, *and Converges very ſoon. So that if* z *be the firſt Minute of the Quadrant, the firſt Term of the Series gives the ſecond Difference to* 15 *Places of Figures, and the ſecond Term to* 25 *Places.*

From hence, if the Sines of the Arcs diſtant one Minute from each other be given. The Sines of all the Arcs, that are in the ſame Progreſſion may be found by an exceeding eaſy Operation.

In the firſt and ſecond Series, if $A=0$; *then ſhall* $a=0$, *and* b *its Coſine, will become Radius, or* 1. *And hence, if the Terms wherein* a *is, are taken away, and* 1 *be put inſtead of* b, *the Series, will become the* Newtonian. *In the third and fourth Series, if* A *be* 90 *Degrees, we ſhall have* $b=0$, *and* $a=1$. *Whence again, taking away all the Terms wherein* b *is, and putting* 1 *inſtead of* a, *we ſhall have the* Newtonian *Series ariſe.*

Note, *All the ſaid Series eaſily flow from the* Newtonian *ones. By the fifth* Propoſition.

PROPOSITION XI.

In a Right-angled Triangle, if the Hypothenuſe be made the Radius, then are the Sides the Sines of their oppoſite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of it's oppoſite Angle, and the Hypothenuſe is the Secant of that Angle.

IT is manifeſt that C B is the Sine of the Arc C D, and A B the Coſine thereof; but the Arc C D is the Meaſure of the Angle A and the Complement of the Meaſure of the Angle C. Moreover, if A B, in the Figure to this Propoſition, be ſuppoſed Radius, then B C is the Tangent, and A C the Secant of the Arc B D, which is the Meaſure of the Angle A. So alſo if B C be made the Radius, then is B A the Tangent, and A C the Secant of the Arc BE, or Angle C. W. W. D. Therefore as A C being taken as ſome given Meaſure is to B C taken in the ſame Meaſure; ſo ſhall the Number 10000000 Parts in which the Radius is ſuppoſed to be divided, be to a Number expreſſing in the ſame Parts the Length of the Sine of the Angle A; that is, it will be.

as A C : B C : : R : S, A.
by the ſame Reaſon, as A C : B A : : R : S, C.
alſo as A B : B C : : R : T, A.
and B C : B A : : R : T, C.

And ſo if any three of theſe Proportionals be given, the fourth may be found by the *Rule of Three.*

PROPOSITION XII.

The Sides of Plain Triangles are as the Sines of their oppoſite Angles.

IF the Sides of a Triangle inſcribed in a Circle be biſected by perpendicular Radii, then ſhall the half Sides be the Sines of the Angles at the Periphery; for the Angle B D C at the Center is double of the Angle B A C at the Periphery; (by 20 *El. lib.* 3.) and ſo the half of every of them, *viz,* $BDE = BAC$, and B E is the Sine thereof. For the ſame Reaſon, B F ſhall be the Sine of the Angle B C A, and A G the Sine of the Angle A B C.

In a Right-angled Triangle we have B D $= \frac{1}{2} BC =$ Radius (*by* 31 *Eucl.* 3.) but Radius

dius is the Sine of a Right Angle: Whence $\frac{1}{2}$ BC is the Sine of the Angle A.

In an Obtuſe-angled Triangle, let BL, CL, be drawn, and then the Angle L ſhall be the Complement of the Angle A to two Right Angles, (*by* 22 *El.* 3.) and ſo they ſhall both have the ſame Sine. But the Angle BDE, (whoſe Sine is BE) = Angle L. Therefore BE ſhall be the Sine of the Angle BAC. And ſo in every Triangle, the Halves of the Sides are the Sines of the oppoſite Angles; but it is manifeſt that the Sides are to one another as their Halves. W. W. D.

PROPOSITION XIII.

In a plain Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Baſe, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whoſe Legs are AB, BC, and Baſe AC. Produce AB to H, ſo that BH = BC, then ſhall AH be the Sum of the Legs; and if you make BI = BA, then IH will be the Difference of the Legs. Alſo the Angle HBC = Angles A + ACB, (*by* 32. *El.* 1.) and ſo EBC the half thereof = half the Sum of the Angles A and ACB, and it's Tangent (putting the Radius = EB) is EC. Again, let BD be drawn parallel to AC, and make HF = CD. Then ſince HB = CB, we ſhall have (*by* 4 *El.* 1.) the Angle HBF = CBD = BCA. (*by* 29 *El.* 1.) Alſo the Angle HBD =

$=$ Angle A; whence F B D ſhall be the Difference of the Angles A and A C B; and E B D, whoſe Tangent is E D, half their Difference. Let I G be drawn thro' I parallel to A C or B D, and then (*by* 2 *El.* 6.) $AB : BI :: CD : DG$, but $AB = BI$; whence we ſhall have $CD = DG$, but $CD = HF$, and ſo $HF = DG$, and conſequently, $HG = DF$, and $\frac{1}{2}HG = \frac{1}{2}DF = DE$; and becauſe the Triangles A H C, I H G, are equiangular, it ſhall be as $AH : IH :: HC : HG :: \frac{1}{2}HC : \frac{1}{2}HG :: EC : ED$. That is, A H the Sum of the Legs, to I H the Difference of the Legs, ſhall be as E C the Tangent of one half the Sum of the Angles at the Baſe, to E D the Tangent of one half their Difference. W. W. D.

PROPOSITION XIV.

In a plain Triangle, the Baſe, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Baſe, are proportional.

LET D C be the Baſe of the Triangle B C D, about the Center B, with the Radius B C, let a Circle be deſcribed. Produce D B to G, and from B let fall B E perpendicular to the Baſe; then ſhall $DG = DB + BC =$ Sum of the Sides, and $DH =$ Difference of the Sides; and D E, C E, are the Segments of the Baſe whoſe Difference is D F; becauſe (*by Cor. Prop.* 38. *El.* 3.) the Rectangle under D C and D F, is equal to the Rectangle under D G, D H, it ſhall be (*by* 16 *El.* 6.) as DC : DG :: DH : DF. W. W. D.

PROBLEM.

The Sum and Difference of any two Quantities being given, to find the Quantities themſelves.

IF one half of the Sum be added to one half of the Difference, the Aggregate ſhall be equal to the greater of the Quantities; and if from one half of the Sum be taken one half of the Difference, the Reſidue ſhall be equal to the leſſer of the Quantities. For let there be two Quantities A B, B C, and let there be taken $AD = BC$, then D B will be their Difference, and A C their Sum; which, biſected in E, gives A E or E C the half Sum, and D E or E B the half Difference. Hence $AB = AE + EB =$ the half Sum $+$ the half Difference, and $BC = CE - EB =$ the half Sum $-$ the half Difference.

In any plain Triangle if two Angles be given, the third Angle is alſo given, becauſe it is their Complement to two Right Angles.

If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, becauſe it is the Complement of the given Angle to a Right Angle.

And if two Sides of a Right-angled Triangle be given, the other Side may be found by the firſt Proportion without a Canon.

The Trigonometrical Solutions of a Right-angled Triangle, may be as follow. Vid. Fig. A.

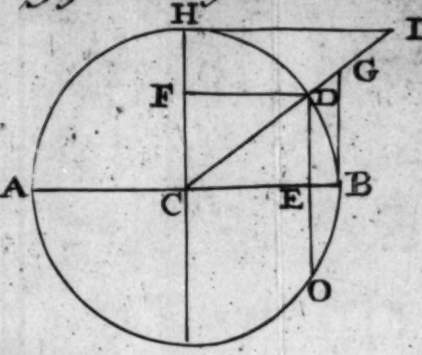
	Given	Sought	Make as
1	The Legs A B and B C	The Angles.	A B: B C : : R: T of the Angle A, whoſe Complement is the Angle C.
2	The Leg A B and the Hypothenuſe A C.	The Angles.	A C: A B: : R: S, C whoſe Complement is the Angle A.
3	The Leg A B and the Angle A.	The other Side B C, and the Hypothenuſe AC	R : T. A : : A B: B C. S, C : R : : A B: A C.
4	The Hypothenuſe A C, and the Angle C.	The Leg A B.	R : S, C : : A C: A B.

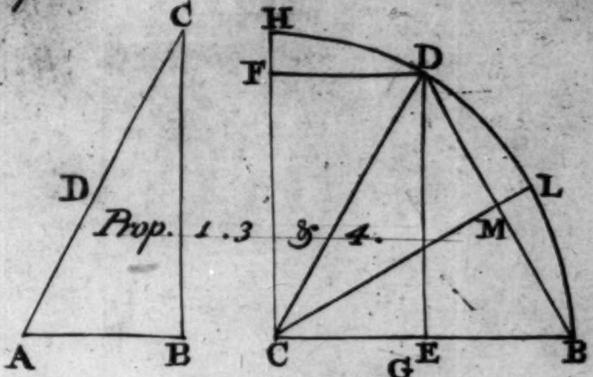
The Trigonometrical Solutions of oblique-angled Triangles. Vid. Fig. to Prop. 12.

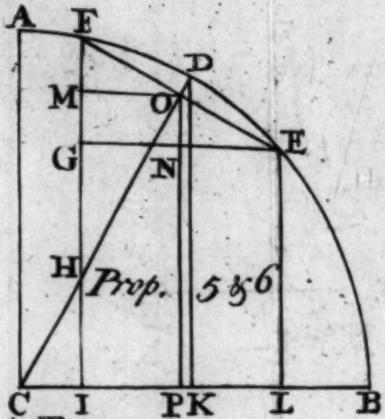
	Given	Sought	Make as
1	The Angles A, B, C, and the Side A B.	The Sides BC and A C.	S, C: S A:: A B: B C. Alſo S, C: S, B:: A B: A C: But when two Angles are given, the third is alſo given; whence the Caſe wherein two Angles and a Side are given, to find the reſt, falls into this Caſe.
2	All the Angles A, B, C,	All the Sides A B, A C, B C.	S, C: S, A :: A B: B C. And S, C: S, B:: A B: B C: Whence if the Angles are given, the Proportions of the Sides may be found, but not the Sides themſelves, unleſs one of them be firſt known.
3	The two Sides A B, BC, and C the Angle oppoſite to one of them.	The Angles A and B.	A B: B C:: S, C: S, A; which therefore may be found. When A B the Side oppoſite to C, the given Angle is longer than B C the Side oppoſite to the ſought Angle, the ſought Angle is leſs than a right one. But when it is ſhorter, becauſe the Sine of an Angle, and that of its Complement to two Right Angles, is the ſame, the Species of the Angle A muſt be firſt known, or the Solution will be ambiguous.

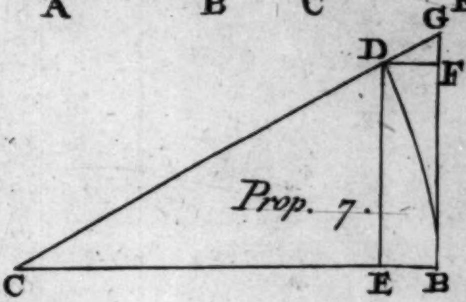
Fig
A
A
C
A
F
G
C
A

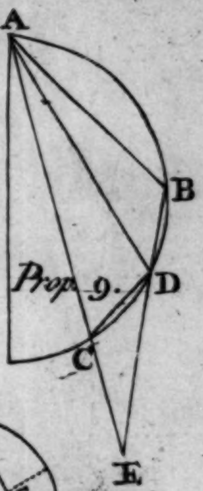
Fig. for the Definitions and Propo 2.

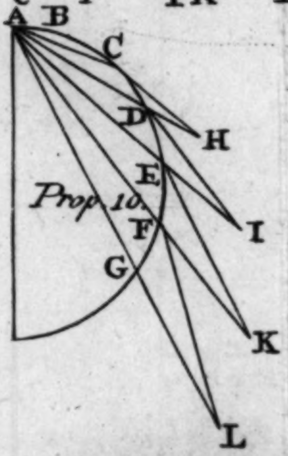


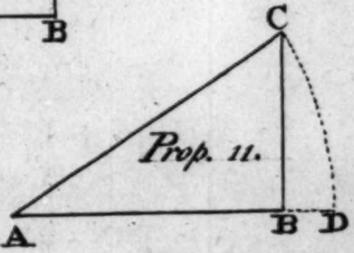

Prop. 1.3 & 4.

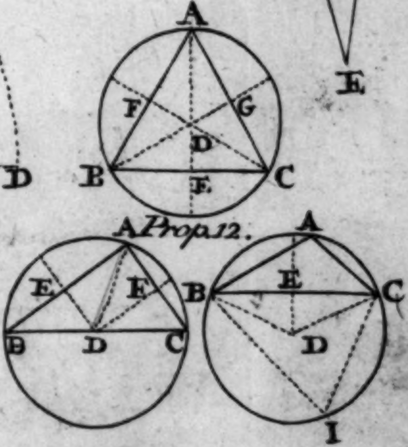

Prop. 5 & 6

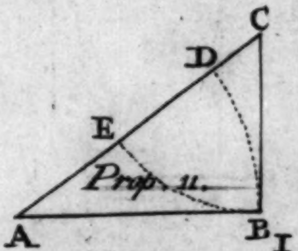

Prop. 7.

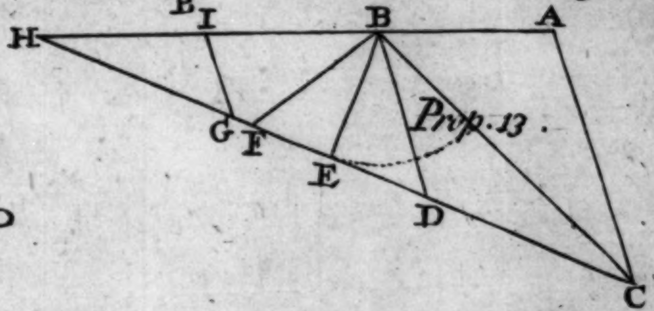

Prop. 9.

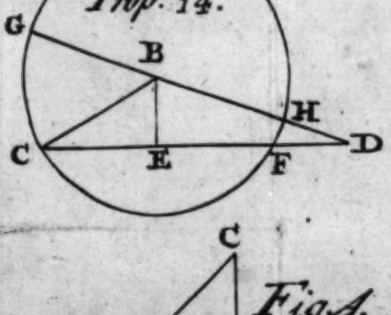

Prop. 10.


Prop. 11.


Prop. 12.


Prop. 11.


Prop. 13.


Prop. 14.

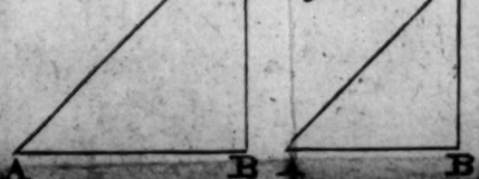


Fig. A.

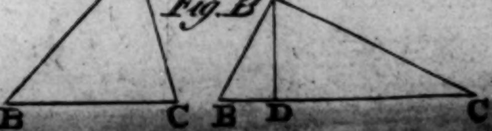


Fig. B

	Given	Sought	Make as
4	The two Sides A B, B C, and the interjacent Angle B.	The Angles A and C.	*Vid. Fig.* to *Prop.* 13. B C + A B : B C − A B :: T, $\frac{A+C}{2}$: T, $\frac{A-C}{2}$ Whence is known the Difference of the Angles A and C, whoſe Sum is given; and ſo (by the *Prob.* following *Prop.* 14.) the Angles themſelves will be given.
5	All the Sides A B, B C, A C.	The Angles.	*Vid. Fig.* B. Let the Perpendicular be drawn from the Vertex to the Baſe, and find the Segments of the Baſe by *Prop.* 14. *viz.* Make as B C : A C + A B :: A C − A B : D C − D B. And ſo B D, D C, are given from this Analogy; and thence the Angles A B D, A D C, will be given by the Reſolution of Right-angled Triangles.

THE ELEMENTS OF Spherical TRIANGLES.

DEFINITIONS.

I. *THE Poles of a Sphere are two Points in the Superficies of the Sphere that are the Extremes of the Axis.*

II. *The Pole of a Circle in a Sphere, is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle, are equal to one another.*

III. *A great Circle in a Sphere, is that whoſe Plane paſſes thro' the Center of the Sphere, and whoſe Center is the ſame as that of the Sphere.*

IV. *A ſpherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

V. *A ſpherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the ſaid Circles.*

* * *

PROPOSITION I.

Great Circles A C B, A F B, mutually biſect each other.

FOR ſince the Circles have the ſame Center, their common Section ſhall be a Diameter of each Circle, and ſo will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being leſs than Semicircles, do not comprehend a Space; for they cannot, unleſs they meet each other in two oppoſite Points in a Semicircle.

PROPOSITION II.

If from the Pole C of any Circle A F B, be drawn a Right Line C D to the Center thereof, the ſaid Line will be perpendicular to the Plane of that Circle. Vid. Fig. to Prop. 1.

LET there be drawn any Diameters E F, G H, in the Circle A F B; then becauſe

in

in the Triangles C D F, C D E, the Sides C D, D F, are equal to the Sides C D, D E, and the Baſe C F, equal to the Baſe C E; (*by Def.* 2.) then (*by* 4 *El.* 1.) ſhall the Angle C D F = Angle C D E; and ſo each of them will be a Right Angle. After the ſame Manner we demonſtrate that the Angles C D G, C D H, are Right Angles; and ſo (*by* 4 *El.* 11.) C D ſhall be perpendicular to the Plane of the Circle A F E. W. W. D.

Coroll. 1. A great Circle is diſtant from it's Pole by the Interval of a Quadrant; for ſince the Angles C D G, C D F, are Right Angles, the Meaſures of them, *viz.* the Arcs C G, C F, will be Quadrants.

2. Great Circles that paſs thro' the Pole of ſome other Circle, make Right Angles with it; and contrariwiſe, if great Circles make Right Angles with ſome other Circle, they ſhall paſs thro' the Poles of that other Circle, for they muſt neceſſarily paſs thro' the Right Line D C.

PROPOSITION III.

If a great Circle E C F be deſcribed about the Pole A; then the Arc C F intercepted between A C, A F, is the Meaſure of the Angle C A F, or C D F, Vid. Fig. to Prop. 1.

THE Arcs A C, A F, (by *Cor.* 1. *Prop.* 2.) are Quadrants, and conſequently the Angles A D C, A D F, are Right Angles. Wherefore (by *Def.* 6. *El.* 11.) the Angle C D F, CDF, (whoſe Meaſure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and alſo equal to the ſpherical Angle CAF, or CBF. W. W. D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, then ſhall A be the Pole of the Circle paſſing thro' the Points C and F; for AD, is at Right Angles to the Plane FDC. (by 14 *El.* 11.)

2. The Vertical Angles are equal, for each of them is equal to the Inclination of the Circles; alſo the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles ſhall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehended by the two Sides alſo equal.

PROPOSITION V.

Alſo Triangles ſhall be equal and congruous, if one Side together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually Equilateral, are alſo mutually equiangular.

PROPO-

PROPOSITION VII.

In Iſoſceles Triangles, the Angles at the Baſe are equal.

PROPOSITION VIII.

And if the Angles at the Baſe be equal, then the Triangle ſhall be Iſoſceles.

Theſe four laſt Propoſitions are demonſtrated in the ſame Manner, as in plain Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle, is the ſhorteſt Way between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a ſpherical Triangle is leſs than a Semicircle.

LET A C, A B, the Sides of the Triangles A B C be produced till they meet in D; then ſhall the Arc A C D, which is greater than the Arc A C, be a Semicircle.

PROPOSITION XI.

The three Sides of a ſpherical Triangle are leſs than a whole Circle.

FOR DB+DC is greater than BC, (by *Prop. 9.*) and adding on each Side BA+AC, DBA+DCA, that is, a whole Circle will be greater than AB+BC+AC, which are the three Sides of the ſpherical Triangle ABC.

PROPOSITION XII.

In any ſpherical Triangle ABC, *the greater Angle* A *is ſubtended by the greater Side.*

MAKE the Angle BAD = Angle B; then ſhall AD = BD, (*by* 8 *of this*;) and ſo BDC = DA+DC, and theſe Arcs are greater than AC. Wherefore the Side BC, that ſubtends the Angle BAC, is greater than the Side AC, that ſubtends the Angle B.

PROPOSITION XIII.

In any ſpherical Triangle ABC, *if the Sum of the Legs* AB *and* BC *be greater, equal, or leſs, than a Semicircle, then the internal Angle at the Baſe* AC *ſhall be greater, equal, or leſs, than the external and oppoſite Angle* BCD, *and ſo the Sum of the Angles* A *and* ACB *ſhall alſo be greater, equal, or leſs, than two Right Angles.*

FIRST, let AB+BC = Semicircle = AD, then ſhall BC = BD, and the Angles B C D and D equal, (*by* 8 *of this*,) and therefore the Angle B C D ſhall be = Angle A.

Secondly, Let A B + B C be greater than A B D; then ſhall B C be greater than B D; and ſo the Angle D (that is, the Angle A, by 12 of this) ſhall be greater than the Angle B C D. In like Manner we demonſtrate, if A B + B C be together leſs than a Semicircle, that the Angle A will be leſs than the Angle B C D. And becauſe the Angles B C D and B C A, are = two Right Angles; if the Angle A be greater than the Angle B C D, then ſhall A and B C A, be greater than two Right Angles; if the Angle A = B C D, then ſhall A and B C A be equal to two Right Angles. And if A be leſs than B C D, then will A and B C A be leſs than two Right Angles. W. W. D.

PROPOSITION XIV.

In any ſpherical Triangle G H D, *the Poles of the Sides being joined by great Circles, do conſtitute another Triangle* X M N, *which is the Supplement of the Triangle* G H D, *viz. the Sides* N X, X M, *and* N M, *ſhall be Supplements of the Arcs that are the Meaſures of the Angles* D, G, H, *to the Semicircles; and the Arcs that are the Meaſures of the Angles* M, X, N, *will be the Supplements of the Sides* G H, G D, *and* H D, *to Semicircles.*

FROM the Poles G, H, D, let the great Circles X C A M, T M N O, X K B N, be deſcribed; then becauſe G is the Pole of the Circle X C A M, we ſhall have G M = Quadrant, (by *Cor.* 1 *Prop.* 2.) and ſince H is the Pole of the Circle T M O, then will H M be alſo a Quadrant; and ſo (by *Cor.* 1. *Prop.* 3.) M ſhall be the Pole of the Circle G H. In like Manner, becauſe D is the Pole of the Circle X B N, and H the Pole of the Circle T M N, the Arcs D N, H N, will be Quadrants; and ſo (by *Cor.* 1. *Prop.* 3.) N ſhall be the Pole of the Circle H D. And becauſe G X, D X, are Quadrants, X will be the Pole of the Circle G D. Theſe Things premiſed.

Becauſe N K = Quadrant, (*Cor.* 1. *Prop.* 2.) then will N K + X B, that is, N X + K B = two Quadrants, or a Semicircle; and ſo N X is the Supplement of the Arc K B, or of the Meaſure of the Angle H D G to a Semicircle. In like Manner, becauſe M C = Quadrant, and X A = Quadrant, then will M C + X A; that is, X M + A C = two Quadrants, or a Semicircle; and conſequently X M is the Supplement of the Arc A C, which is the Meaſure of the Angle H G D. Likewiſe, ſince M O, N T, are Quadrants, we ſhall have M O + N T = O T + N M = Semicircle. And therefore N M is the Supplement of the Arc O T, or of the Meaſure of the Angle G H D, to a Semicircle. W. W. D.

Moreover, becauſe D K, H T, are Quadrants, D K + H T, or K T + H D, are equal to two Quadrants, or a Semicircle. Therefore K T, or the Meaſure of the Angle X N M, is the Supplement of the Side H D to a Semicircle.

micircle. After the ſame Manner it is demonſtrated, that O C, the Meaſure of the Angle X M N, is the Supplement of the Side G H, and B A the Meaſure of the Angle X, is the Supplement of the Side G D. W. W. D.

PROPOSITION XV.

Equiangular ſpherical Triangles are alſo equilateral.

FOR their Supplementals (*by* 14 *of this*) are equilateral; and therefore equiangular alſo; and ſo themſelves are likewiſe equilateral, (by *Part* 2. *Prop.* 14.)

PROPOSITION XVI.

The three Angles of a ſpherical Triangle, are greater than two Right Angles, and leſs than ſix.

FOR the three Meaſures of the Angles G, H, D, together with the three Sides of the Triangle X N M, make three Semicircles, (*by* 14 *of this*;) but the three Sides of the Triangle X N M, are leſs than two Semicircles, (*by* 11 *of this.*) Wherefore the three Meaſures of the Angles G, H, D, are greater than a Semicircle; and ſo the Angles G, H, D, are greater than two Right Angles.

The ſecond Part of the Propoſition is manifeſt, for in every ſpherical Triangle, the external and internal Angles together, only make ſix Right Angles; wherefore the internal Angles are leſs than ſix Right Angles.

PROPO-

PROPOSITION XVII.

If from the Point R, *not being the Pole of the Circle* A F B E, *there fall the Arcs* R A, R B, R G, R V, *of great Circles to the Circumference of that Circle; then the greateſt of thoſe Arcs is* R A, *which paſſes thro' the Pole* C *thereof; and the Remainder of it is the leaſt; and thoſe that are more diſtant from the greateſt are leſs than thoſe which are nearer to it, and they make an obtuſe Angle with the former Circle* A F B, *on the Side next to the greateſt Arc.* Vid. Fig. to Prop. 1.

BEcauſe C is the Pole of the Circle A F B, then ſhall C D and R S, which is parallel thereto, be perpendicular to the Plane A F B. And if S A, S G, S V, be drawn, then ſhall S A (by 7 *El.* 3.) be greater than S G, and S G greater than S V. Whence in the Right-angled plain Triangles R S A, R S G, R S V, we ſhall have $RSq + SAq$, or RAq, greater than $RSq + SGq$, or RGq; and ſo R A will be greater than R G, and the Arc R A greater than the Arc R G. In like Manner, $RSq + SGq$, or RGq ſhall be greater than $RS + SVq$, or RVq; and ſo R G ſhall be greater than R V, and the Arc R G greater than the Arc R V.

2dly, The Angle R G A is greater than the Angle C G A, which is a Right Angle (by *Cor. Prop.* 3.) and the Angle R V A is greater than the Angle C V A, which alſo is a Right Angle. Therefore the Angles R G A, R V A, are obtuſe Angles.

PROPOSITION XVIII.

In a ſpherical Triangle right-angled at A, *the Legs containing the Right Angle, are of the ſame Affection with the oppoſite Angles; that is, if the Legs be greater or leſs than Quadrants, then accordingly will the Angles oppoſite to them be greater or leſs than Right Angles.* Vid. Fig. to Prop. 1.

FOR if A C be a Quadrant, then will C be the Pole of the Circle A F B, and the Angles A G C, or A V C, will be Right Angles. If the Leg A R be greater than a Quadrant, then ſhall the Angle A G R be greater than a Right Angle, (*by* 17 *of this*;) and if the Leg A X be leſs than a Quadrant, the Angle A G X ſhall be leſs than a Right Angle.

PROPOSITION XIX.

If two Legs of a right-angled ſpherical Triangle be of the ſame Affection; (and conſequently the Angles,) that is, if they are both leſs or both greater than a Quadrant, then will the Hypothenuſe be leſs than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle A R V, or B R V, let F be the Pole of the Leg A R, then will R F be a Quadrant, which is greater than R V, (*by* 17 *of this.*)

PROPO-

PROPOSITION XX.

If they be of a different Affection, then ſhall the Hypothenuſe be greater than a Quadrant. Vid. Fig. to Prop. 1.

FOR in the Triangle A R G, the Hypothenuſe R G is greater than R F, which is a Quadrant.

PROPOSITION XXI.

If the Hypothenuſe be greater than a Quadrant, then the Legs of the Right Angle, and ſo the Angles oppoſite to them, are of different Affection; but if leſſer, of the ſame Affection. Vid. Fig. to Prop. 1.

THIS Propoſition being the Converſe of the former ones, eaſily follow from them.

PROPOSITION XXII.

In any ſpherical Triangle A B C, *if the Angles at the Baſe* B *and* C, *be of the ſame Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.*

IN the firſt Caſe, if the Perpendicular does not fall within, let it fall without the Triangle, (as in *Fig.* 2.) then in the Triangle A B P, the Side A P is of the ſame Affection with the Angle B. And in like Manner, in the Triangle A C P, A P, is of the ſame Affection with the Angle A C P. Therefore ſince A B C, and A C P, are of the ſame Affection, the Angles A B C, A C B, ſhall be of different Affection; which is contrary to the Hypotheſis.

In the ſecond Caſe, if the Perpendicular does not fall without, let it fall within, (as in *Fig.* 1.) Then in the Triangle A B P, the Angle B is of the ſame Affection with the Leg A P. So likewiſe, in the Triangle A C P, the Angle C is of the ſame Affection with A P; and therefore the Angles B and C are of the ſame Affection; which is contrary to the Hypotheſis.

PROPOSITION XXIII.

In ſpherical Triangles B A C, B H E, *right-angled at* A *and* H, *if the ſame acute Angle* B *be at the Baſe* B A, *or* B H, *then the Sines of the Hypothenuſe ſhall be proportional to the Sines of the perpendicular Arcs.*

FOR the Right Lines C D, E F, being perpendicular to the ſame Plane, are parallel. Alſo F R, D P, perpendicular to the Radius O B, are likewiſe parallel; wherefore the Planes of the Triangles E F R, C D P, are alſo parallel, (by 15 *El.* 11.) Wherefore C P, E R, the common Sections of thoſe Planes, with the Plane paſſing thro' B E, C O, will be parallel, (by 16 *El.* 11.) Therefore the Triangles C D P, E F R, ſhall be equiangular. Wherefore C P the Sine of the Hypothenuſe B C, is to C D the Sine of the perpendicular Arc C A, as E R, the Sine of the Hypothenuſe B E to E F, the Sine of the perpendicular Arc E H. W. W. D.

PROPOSITION XXIV.

The ſame Things being ſuppoſed, A Q, H K, *the Sines of the Baſes, are proportional to* I A, G H, *the Tangents of the perpendicular Arcs.*

FOR after the ſame Manner, as in the laſt Propoſition, we demonſtrate that the Triangles Q A I, K H G, are equiangular; whence Q A : A I :: K H : H G.

PROPOSITION XXV.

In a ſpherical Triangle A B C, *right-angled at* A, *as the Coſine of the Angle* B, *at the Baſe* B A, *is to the Sine of the vertical Angle* A C B, *ſo is the Coſine of the Perpendicular to the Radius.*

PREPARATION.

LET the Sides A B, B C, C A, be produced, ſo that B E, B F, C I, C H, be Quadrants; and from the Poles B and C, draw the great Circles E F D G, I H G, then will the Angles at E, F, I, H, be Right Angles. And ſo D is the Pole of B A E, (by *Cor.* 2. *Prop.* 2 of this,) and G the Pole of I F C B; alſo A E will be = Complement of the Arc B A, and F E the Meaſure of the Angle B = G D, and D F their Complement. : Alſo B C ſhall be = F I = Meaſure of the Angle G, and C F their Complement. Likewiſe C A = H D, and D C their Complement. Theſe Things premiſed in the Triangles H I C, D C F, D C F, right-angled at I and F, and having the ſame acute Angle C, ſince B A is leſs than a Quadrant, it will be as S, D F : S, H I : : S, D C : S, H C; that is, the Coſine of the Angle B, is to the Sine of the vertical Angle B C A, as the Coſine of C A is to Radius. W. W. D.

PROPOSITION XXVI.

The Coſine of the Baſe : *Coſine of the Hypothenuſe* : : R : *Coſ. of the Perpendicular.*

FOR in the Triangles A E D, C F D, right-angled at E, F, having the ſame acute Angle D; becauſe A E is leſs than a Quadrant, we have S, E A : S, C F : : S, D A : S, D C. W. W. D.

PROPOSITION XXVII.

S, *of the Baſe* : R : : T, *of the Perpendicular* : T, *of the Angle at the Baſe.*

FOR in the Triangles B A C, B E F, right-angled at A and E, and having the ſame acute Angle B; becauſe A C is leſs than a Quadrant, we have S, B A : S, B E : : T, A C : T, E F. W. W. D.

PROPOSITION XXVIII.

Coſ. of the vertical Angle : R : : T, *of the Perpendicular* : T, *of the Hypothenuſe.*

IN the Triangles G I F, G H D, right-angled at I and H, and having the ſame acute Angle G, becauſe H D is leſs than H C, or a Quadrant, it is as S, G H : S, G I : : T, H D : T, I F.

PROPOSITION XXIX.

S, *of the Hypothenuſe* : R : : S, *of the Perpendicular* : S, *of the Angle at the Baſe.*

IN the aforeſaid Triangles, we have S, I F : S, G F : : S, H D : S, G D.

PROPOSITION XXX.

R : *Coſ. of the Hypothenuſe* : : T, *of the vertical Angle* : *Cot. of the Angle at the Baſe.*

IN the Triangles H I C, D F C, right-angled at I and F, and having the ſame acute Angle C, becauſe D F is leſs than a Quadrant, we have S, C I : S, C F : : T, H I : T, D F, that is, R : Coſ. B C : : T, C : Cot. B.

The laſt ſix Propoſitions are ſufficient for ſolving all the ſixteen Caſes of Right-angled ſpherical Triangles. Theſe ſixteen Caſes, with their Analogies deduc'd from the ſaid Propoſitions, are as follow :

	Given beſides the Right Angle.	Sought	The Proportions. *Vid. Fig.* to *Prop.* 25, 26, 27, 28, 29, 30.	
1	A C and C.	B	R : Coſ. C A : : S, C : Coſ. B of the ſame Kind with C A.	By the Inverſe of Prop. 25.
2	A C and B.	C.	Coſ. C A : R : : Coſ. B : S, C this is ambiguous.	by Prop. 25.
3	B and C.	A C.	S, C : Coſ. B : : R : Coſ. C A of the ſame kind with the Angle B.	by Prop. 25 and 18.
4	B A, C A.	B C.	R : Coſ. B A : : Coſ. A C : Coſ. B C. If B A, A C, be of the ſame Affection, and not Quadrants, then B C will be leſs than a Quadrant. If they be of a different Affection, B C ſhall be greater than a Quadrant.	by Prop. 26. 19, and 20.
5	B A, B C.	A C.	Coſ. B A : R : : Coſ. B C : Coſ. C A. If B C be leſs than a Quadrant, then ſhall B A and C A be of the ſame Affection; if greater, of a different: but B A is given, and therefore the Species thereof: Wherefore the Species of A C is alſo given.	by Prop. 26 and 21.
6	B A, C A.	B.	S, B A : R : : T, C A : T, B of the ſame Affection with the oppoſite Side C A.	by Prop. 27 and 18.

Given

	Given beſides the Right Angle.	Sought	The Proportions.	
7	B A, B.	A C.	R : S, B A : : T, B : T, A C of the ſame kind with B.	by Prop. 27 and 18.
8	A C, B.	B A.	T, B : R : : T, C A : S, B A ambiguous.	by Prop. 27.
9	BC, C.	A C.	R : Coſ. C : : T, B C : T, C A. If B C be leſs than a Quadrant, the Angles C and B are of the ſame Affection, if greater, of a different. Therefore if the Species of the Angle B be given, then will A C be given.	By Prop. 28 and 21.
10	AC, C.	B C.	Coſ. C : R : : T, C A : T, B C. And ſo if the Angle C, and C A, be of the ſame Affection, then B C ſhall be leſſer than a Quadrant, if of a different, greater.	by Prop. 28, 20, 21.
11	B C, A C.	C.	T, B C : R : : T, C A : Coſ. C. If B C be leſs than a Quadrant, then C A and B A, and conſequently the Angles, ſhall be of the ſame Affection, if greater, of a different, but the Species of C A is given. Therefore the Species of the Angle C will be alſo given.	by Prop. 28, and 21.

Given

	Given beſides the Right Angle.	Sought	The Proportions.	
12	BC, B.	A C.	R : S, BC : : S, B : S, A C of the ſame Species with B.	by Prop. 29, and 18.
13	AC, B.	B C.	S, B : S, A C : : R : S, B C ambiguous.	by Prop. 29.
14	B C. A C.	B.	S, B C : R : : S, A C : S, B of the ſame Species with CA.	by Prop. 29.
15	B, C.	B C.	T, C : R : : Cot. B : Coſ. B C. And ſo if the Angles B and C are of the ſame Affection, then ſhall B C be leſs than a Quadrant, if of a different, greater.	by Prop. 30, 19, and 20.
16	BC, C.	B.	R : Coſ. B C : : T, C : Cot. B. And ſo if B C be leſſer than a Quadrant, the Angles C and B ſhall be of the ſame Affection ; if greater of a different. But the Species of the Angle C is given ; therefore the Species of the Angle B will be given alſo.	by Prop. 30, and 31.

Of the Solution of Right-angled Spherical Triangles, by the five Circular Parts.

THE Lord *Napier*, (the Noble Inventor of Logarithms,) by a due Conſideration of the Analogies, by which Right-angled ſpherical Triangles are ſolv'd, found out two Rules, eaſy to be remembred, by means of which, all the ſixteen Caſes may be ſolved; for ſince, in theſe Triangles, beſides the Right Angle, there are three Sides and two Angles; the two Sides comprehending the Right Angle, and the Complements of the Hypothenuſe, and the two other Angles, were called by *Napier*, *Circular Parts*. And when there are given any two of the ſaid Parts, and a third is ſought; one of theſe three which is called the *Middle-Part*, either lyes between the other two Parts, which are called *Adjacent-Extremes*, or is ſeparated from them, and then are called *Oppoſite-Extremes*; ſo if the Complement of the Angle B (*Fig.* to *Prop.* 25.) be ſuppoſed the middle Part, then the Leg A B, and the Complement of the Hypothenuſe B C, are adjacent Extreme Parts; but the Complement of the Angle C, and the Side A C, are oppoſite Extremes. Alſo if the Complement of the Hypothenuſe B C, be ſuppoſed the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs A B, A C, are oppoſite Extremes. In like Manner, ſuppoſing the Leg A B the middle Part, the Complement of the Angle B and A C, are adjacent Extremes; for the Right Angle A does not interrupt the Adjacence, becauſe it is not a Circular Part. But the Complement of the Angle C, and the Complement of the Hypothenuſe B C, are oppoſite Extremes to the ſaid middle Part. Theſe Things premiſed.

RULE. I.

In any Right-angled ſpherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

RULE. II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Coſines of the oppoſite Parts.

Each of the Rules have three Caſes. For the middle Part may be the Complement of the Angle B, or C, or the Complement of the Hypothenuſe B C; or one of the Legs, A B, A C.

Caſe 1. Let the Complement of the Angle C be the middle Part. Then ſhall A C, and the Complement of the Hypothenuſe B C, be adjacent Extremes. By *Prop.* 28. the Coſine of the Vertical Angle C is to Radius, as the Tangent of C A, is to the Tangent of the Hypothenuſe B C. Then (by Alternation) we ſhall have Coſ. C: T, CA: : R: T,BC. But R: T, BC: : Cot. BC: R. (as has been before ſhewn.) Wherefore Coſ. C: T, AC: : Cot. BC: R; whence R × Coſ. C = T, AC × Cot. BC.

And the Complement of the Angle B, and A B, are oppoſite Extremes, to the ſame middle Part, the Complement of the Angle C. (and by *Prop.* 25.) as the Coſine of the Angle C, to the Sine of the Angle C D F, ſo is the Coſine of D F to Radius. But the Sine of C D F = S, A E = Coſ. B A, and Coſ. D F = S, E F = S, Angle B. Whence it will be as Coſ, C: Coſ. B A: : S, B: R. And R × Coſ. C = Coſ. B A × S, B; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Coſines of the oppoſite Extremes.

Caſe 2. Let the Complement of the Hypothenuſe B C, be the middle Part, then the Complements of the Angles B and C, will be adjacent Extremes. In the Triangle D C F (by *Prop.* 27.) it is as S, C F: R: : T, D F. T, C. Whence (by Alternation) S, C F: T, D F: : (R: T, C: :) Cot. C: R. But S, C F = Coſ. B C and T, D F = Cot. B. Wherefore R × Coſ. B C = Cot. C × Cot. B; that is, Radius drawn into the Sine of the middle Part, is equal to the Product of the Tangents, of the adjacent extreme Parts.

And A B, A C, are the oppoſite Extremes to the ſaid middle Part, *viz.* the Complement of B C; and (by Prop. 26.) Coſ. B A: Coſ. B C: :

BC : : R : Coſ. AC. Wherefore we ſhall have R × Coſ. BC = Coſ. BA × Coſ. AC.

Caſe 3. Laſtly, Let AB be the middle Part; and then the Complement of the Angle B, and AC will be adjacent Extremes, (and by Prop. 27.) S, AB : R : : T, CA : T, B. Whence, S, AB : T, CA : : (R : T, B : :) Cot. B : R. And ſo R × S, AB = T, CA × Cot. B.

Moreover, the Complements of BC, and the Angle C, are oppoſite Extremes to the ſame middle Part AB; and in the Triangle GHD (by Prop. 25.) we have Coſ. D : S, DGH : : Coſ. GH : R. But Coſ. D = Coſ. AE = S, AB, and S, G = S, IF = S, BC. Alſo Coſ. GH = S, HI = S, C. Wherefore it will be as S, AB : S, BC : : S, C : R. And hence R × S, AB = S, BC × S, C.

And ſo in every Caſe the Rectangle under the Radius, and the Sine of the middle Part, ſhall be equal to the Rectangle under the Coſines of the oppoſite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes. And conſequently if the aforeſaid Equations be reſolved into Analogies, (by 16. *El.* 6.) the unknown Parts may be found by the Rule of Proportion. And if the Part ſought be the middle one; then ſhall the firſt Term of the Analogy be Radius, and the ſecond and third, the Tangents or Coſines of the extreme Parts. If one of the Extremes be ſought, the Analogy muſt begin with the other; and the Radius, and the Sine of the middle Part, muſt be put in the middle Places, that ſo the Part ſought may be in the fourth Place.

IN the Oblique-angled Spherical Triangle (*Fig. to Prop.* 31.) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Baſe continued, if need be, ſo as to make two Right-angled Spherical Triangles BAC, DAC, then by thoſe Right-angled Triangles, may moſt of the Caſes of Oblique-angled ones be ſolved.

PROPOSITION XXXI.

The Coſines of the Angles B *and* D, *at the Baſe* BD, *are proportional to the Sines of the Vertical Angles* BCA, DCA.

FOr Coſ. Angle B : S, BCA : : (Coſ. CA : R : :) Coſ. D : S, DCA. (*by* 25. *of this.*)

PROPOSITION XXXII.

The Coſines of the Sides BC, DC, *are proportional to the Coſines of the Baſes* BA, DA.

FOR Coſ. BC : Coſ. BA : : (Coſ. CA : R : :) Coſ. DC : Coſ. DA. (*by* 26. *of this.*)

PROPOSITION XXXIII.

The Sines of the Baſes B A, D A, *are in a reciprocal Proportion of the Tangents of the Angles* B *and* D *at the Baſe* B D.

BECAUSE (*by* 27. *of this.*) S, B A : R : : T, A C : T. of the Angle B. And by the ſame inverſely, R : S, D A : : T, of the Angle D : T, A C. Then will it be (by the Equality of perturbate Ratio, according to *Prop.* 23. *El.* 5.) S, B A : S, D A : : T, Angle D : T, Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides B C, D C, *are in a reciprocal Proportion of the Coſines of the Vertical Angles* B C A, D C A.

BECAUSE by alternating the 28th *Prop.* we have

T, B C : R. : : T, C A : Coſ. B C A, and by the ſame R : Coſ. D C A : : T, D C : T, C A. Wherefore by Equality of perturbate Proportion.

T, B C : Coſ. D C A : : T, D C : Coſ. B C A.

PROPOSITION XXXV.

The Sines of the Sides B C, D C, *are proportional to the Sines of the oppoſite Angles* D *and* B.

BECAUSE by the 29th *of this*, S, B C : R : : S, C A : S, of the Angle B. And by the ſame, inverting R : S, D C : : S, Angle D : S, of C A, whence by Equality of perturbate Ratio, S, B C : S, D C : : S, D : S, B.

PROPOSITION XXXVI.

In any Spherical Triangle A B C, *the Rectangle* C F × A E, *or* F M × A E, *contained under the Sines of the Legs* B C, B A, *is to the Square of the Radius, as* I L *or* I A — L A *the Difference of the Verſed Sines of the Baſe* C A, *and the Difference of the Legs* A M, *to* G N, *the Verſed Sine of the Angle* B.

LET a great Circle P N be deſcribed from the Pole B; and let B P, B N be Quadrants, and then P N is the Meaſure of the Angle B; alſo deſcribe from the ſame Pole B, a leſs Circle C F M thro' C; the Planes of theſe Circles ſhall be perpendicular to the Plane B O N, (*by the 2d of this.*) And P G, C H, being perpendicular in the ſame Plane, fall on the common Sections O N, F M; ſuppoſe in G, H. Again draw H I, perpendicular to A O; and then the Plane drawn thro' C H, H I, ſhall be perpendicular to the Plane A O B. Whence A I which is perpendicular to H I will be perpendicular to the Right Line C I, (by *Def.* 4. *El.* 11.) and ſo A I is the Verſed Sine of the Arc A C, and A L the Verſed Sine of the Arc A M = B M — B A = B C — B A. The Iſoſceles Triangles C F M, P O N, are equiangular, ſince M F, N O, as alſo C F, P O, (by 16. *El.* 11.) are parallel. Wherefore, if Perpendiculars C H, P G be drawn to the Sides **FM, ON**, the Triangles will be divided ſimilarly, and we ſhall have **FM : ON :: MH : GN**. Alſo, becauſe the Triangles **AOE, DIH, DLM**, are equiangular, we ſhall have **AE : AO :: IL : MH**.

But it has been proved that **FM : ON :: MH : GN**. Wherefore it ſhall be as **AE × FM : AO × ON :: IL × MH : MH × GN**, or as **IL** to **GN**, that is, the Rectangle under the Sines of the Legs, is to the Square of Radius, as the Difference of the Verſed Sines of the Baſe, and the Difference of the Legs **BC, BA**, is to the Verſed Sine of the Angle **B**. W. W. D.

PROPOSITION XXXVII.

The Difference of the Verſed Sines of two Arcs drawn into half the Radius, is equal to the Rectangle under the Sine of half the Sum, and the Sine of half the Difference of thoſe Arcs.

LET there be two Arcs **BE, BF**, whoſe Difference **EF**, let be biſected in **D**; then ſhall **BD** be the half Sum, and **FD** the half Difference of thoſe Arcs. **GE = IL** is the Difference of the Verſed Sines of the Arcs **BE, BF**; alſo **FO** is the Sine of the half Difference of the Arcs. And becauſe the Triangles **CDK, FEG**, are equiangular, we have **DK : GE :: (CD : FE ::)** $\frac{1}{2}$ **CD** : $\frac{1}{2}$ **FE**. Whence **DK** × $\frac{1}{2}$ **FE**, or **DK × FO** = **GE** × $\frac{1}{2}$ **CD** = **IL** × $\frac{1}{2}$ **CD**. W. W. D.

PROPOSITION XXXVIII.

The Verſed Sine of any Arc, drawn into the half the Radius, is equal to the Square of the Sine of one half of the ſaid Arc.

THE Triangles CBM, DEB are equiangular, ſince the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore EB : BD : : BM : BC. And then will $EB \times BC = BM \times BD$; and $EB \times \frac{1}{2} BC = BM \times \frac{1}{2} BD = BMq$. W. W. D.

PROPOSITION XXXIX.

In any Spherical Triangle ABC, *whoſe Legs containing the Angle* B *are* BC, AB, *and Baſe ſubtending that Angle* AC: *If the Arc* AM *be taken = Difference of the Legs =* BC — AB. *Then ſhall the Rectangle under the Sines of the Legs* BC, BA, *be to the Square of the Radius, as the Rectangle under the Sine of the Arc* $\frac{AC+AM}{2}$, *and the Sine of the Arc* $\frac{AC-AM}{2}$ *is to the Square of the Sine of one half of the Angle* B. Vid. Fig. to Prop. 36.

BECAUSE the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the Verſed Sine of the

Angle B, or as $\frac{1}{2}$ R. × I L to $\frac{1}{2}$ R. drawn into the Verſed Sine of the Angle B (by *Prop.* 36. of this.) And ſince $\frac{1}{2}$ R. × I L = Rectangle under the Sines of the Arcs $\frac{AC + AM}{2}$, and $\frac{AC - AM}{2}$ (by *Prop.* 37. of this.) And alſo $\frac{1}{2}$ R. drawn into the Verſed Sine of the Angle B is equal to the Square of the Sine of one half of the Angle B. Therefore the Rectangle under the Sines of the Sides to the Square of Radius, ſhall be as the Rectangle under the Sines of the Arcs $\frac{AC + AM}{2}$ and $\frac{AC - AM}{2}$, is to the Square of the Sine of one half the Angle B. W. W. D.

The twelve Caſes of Oblique-angled Spherical Triangles are as follow.

	Given.	Sought.	make as
1	Angles B, D, and BC.	Angle C.	R : Coſ. BC :: T, B : Cot. BCA, (by *Prop.* 30. of this.) Alſo Coſ. B : S, BCA :: Coſ. D : S, DCA (by 31. of this.) Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = BCD. Whether the Perpendicular falls within, or without the Triangle, may be known from the Affection of the Angles B and D (by 22. of this,) which Admonition ought to be obſerved in the following Solutions.
2	Angles B, BCD and the Side BC.	Angle D.	R : Coſ. BC :: T, B : Cot. BCA (*Prop.* 30. of this.) And S, BCA : S, DCA :: Coſ. B : Coſ. D (by *Prop.* 31.) If BCA be leſs than BCD, the Angle D ſhall be of the ſame Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D, ſhall be of a different Affection, by the Converſe of *Prop.* 22.

In the Original, the Proportion was thus, *Coſ.* BC : R :: T, B : Cot. BCA.

This Proportion in the Original was as in the foregoing. The Species of the Angle BCA may be known by Prop. 18. *and* 19.

Given

	Given.	Sought	make as
3	The Sides B C, C D and the Angle B.	The Side B D.	R: Coſ. B:: T, BC: T, B A. (by 28, of this.) and Coſ. BC : Coſ. B A :: Coſ. DC : Coſ. D A. (by 32. of this.) the Sum and Difference of BA or DA, according as the Perpendicular falls within, or without the Triangle, is equal to B D, which cannot be known, unleſs the Species of the Angle D be firſt known.
4	The Sides B C, D B and the Angle B.	The Side C D.	R : Coſ. B: : T, BC : T, B A. (by 28 of this.) and Coſ. B A : Coſ. BC : : Coſ. D A : Coſ. DC. (by *Prop.* 32. of this.) According as D A is ſimilar or diſſimilar to C A, or to the Angle B D C, ſo ſhall D C be leſs or greater than a Quadrant, (by 19. and 20 of this.)
5	Angles B, D, and the Side B C.	The Side B D.	R : Coſ. B : : T, BC: T, B A. (by 28. of this.) And T, D: T, B : : S, B A : S, D A. (by 33. of this.) The Sum or Difference of which B A and D A is = B D.
6	The Sides B C, B D and the Angle B.	Angle D.	R : Coſ. B : : T, BC : T, B A, (by *Prop.* 28. of this.) and S, D A : S, B A :: T, B : T, D. (by 33. of this.) According as BD is greater or leſs than B A, the Angle D ſhall be ſimilar or diſſimilar to the Angle B. (by 22. of this.)

	Given.	ſought	make as
7	The Sides B C, D C, and the Angle B.	The Angle C.	Coſ. B C : R : : Cot. B : T, B C A, (by 30. of this.) and T, D C : T, B C : : Coſ. B C A : Coſ. D C A, (by 34. of this.) The Sum or Difference of the Angles B C A, D C A, according as the Perpendicular falls within, or without the Triangle, is equal to the Angle B C D.
8	The Angles B, C, and Side B C.	The Side D C.	Coſ. B C : R : : Cot. B : T, B C A, (by 30. of this.) Alſo Coſ. D C A : Coſ. B C A : : T, B C, T, D C, (by 34. of this.) If the Angle D C A be ſimilar to the Angle B (that is, if A D be ſimilar to C A,) then D C ſhall be leſs than a Quadrant. If the Angles D C A and B be diſſimilar, then D C ſhall be greater than a Quadrant, which follows (from *Prop.* 18, 19, and 20. of this.)
9	The Sides B C, D C and the Angle B.	The Angle D.	S, C D : S, B : : S, B C : S, D, which is Ambiguous. The Analogy follow from *Prop.* 35. of this.
10	The Angles B, D, and the Side B C.	The Side D C.	S, D : S, B C : : S, B : S, D C, which Side is Ambiguous.

	Given.	ſought	make as
11	All the Sides A B, B C, C A. *Vid. Fig. Prop.* 36.	The Angle B.	As the Rectangle under the Sines of the Legs A B, B C : The Square of Radius: : the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B, (by *Prop.* 39.
2	All the Angles G, H, D. *Vid. Fig.* 14.	The Side G D.	In the Triangle X N M, the Arc M N is the Complement of the Angle G H D to a Semicircle. X M is the Complement of the Angle G, and X N the Complement of the Angle D. And the Angle X, the Complement of the Side G D to a Semicircle. Wherefore if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the ſame, as in *Caſe* 11, of this, ſince Arcs and their Complements to Semicircles have the ſame Sines.

The

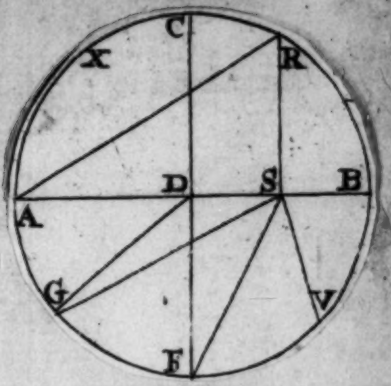
The following REMARK *by* SAMUEL CUN.

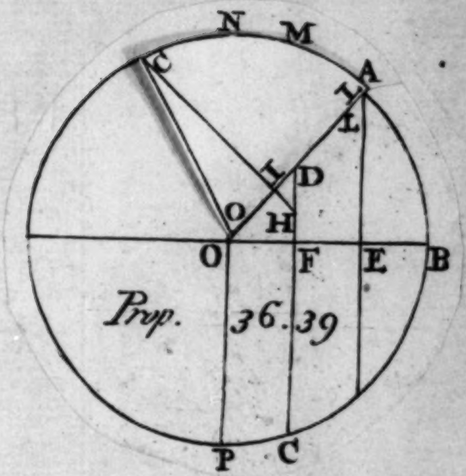
THAT this is true, but in a particular Caſe, *viz.* when two of the Angles of the Triangle are Right ones, and two of the Sides Quadrants, may be thus demonſtrated. For if poſſible, let ſome Triangle R S T, *Fig.* to *Prop.* 14th be ſuch, that its Sides R S, S T, T R, be equal to the Meaſures of G H D, H G D, G D H, the Angles of a Triangle G H D; and alſo, that the Meaſures of R S T, S T R, T R S, the Angles of the Triangle R S T, be equal to G H, G D, H D, the Sides of the Triangle G H D. And produce M X, M N, two Sides of the ſupplemental Triangle to Semicircles, and they will meet ſome-where, ſuppoſe at E; and there will be conſtructed thereby the Triangle N E X, of which X E (the Supplement of X M, which, by the 14th *Prop.* was the Supplement of the Meaſure of the Angle H G D) is equal to the Meaſure it ſelf of the ſame Angle H G D: And in like manner, N E (the Supplement of N M, which, by the 14th *Prop.* was the Supplement of the Meaſure of the Angle G H D) is equal to the Meaſure it ſelf of the ſame Angle G H D. But the third Side X N, is not the Meaſure of the third Angle G D H, but its Supplement, by the 14th *Prop.* Moreover, of the Angle E X N (whoſe Supplement is N X M) the Meaſure, by the 14th *Prop.* is equal to G D; and of the Angle X N E, (whoſe Supplement is M N X) the Meaſure, by the 14th *Prop.* is equal to H D. But of the third N E X, (which is equal to N M X) the Meaſure is not equal to G H, but its Supplement.

Now make N V = R T = B K, the Meaſure of the Angle G D H, and draw the great Circle E V. And ſince R S, by Suppoſition, is equal to the Meaſure of the Angle G H D, which is equal to E N; and ſince the Meaſure of the Angle S R T, is by Suppoſition, equal to D H, which is alſo equal to the Meaſure of the Angle X N E; the Angle X N E, is equal to the Angle R. Then conſequently, by the 4th *Prop.* the Triangles S R T, E N V, will have the Baſe S T, equal to the Baſe E V; the Angle T, to the Angle N V E, and the Angle S, to the Angle N E V. But S T, (which is equal to E V,) by Suppoſition, is equal to the Meaſure of the Angle H G D; to which Meaſure X E is alſo equal. Therefore, E V is equal to X E; and conſequently, by the 7th *Prop.* the Angle E V X is equal to the Angle E X V; and the Angle E X V (whoſe Meaſure, as hath been ſhewn above, is equal to G D) is equal to the Angle T, (or N V E,) ſince by Suppoſition, the Meaſure of this is alſo equal to G D. Therefore the Angle E V X, is equal to the Angle E V N, and ſo both Right ones; and conſequently E X V a Right one alſo. Therefore, by the 2d *Cor.* to the 2d *Prop.* E V and E X are both Quadrants.

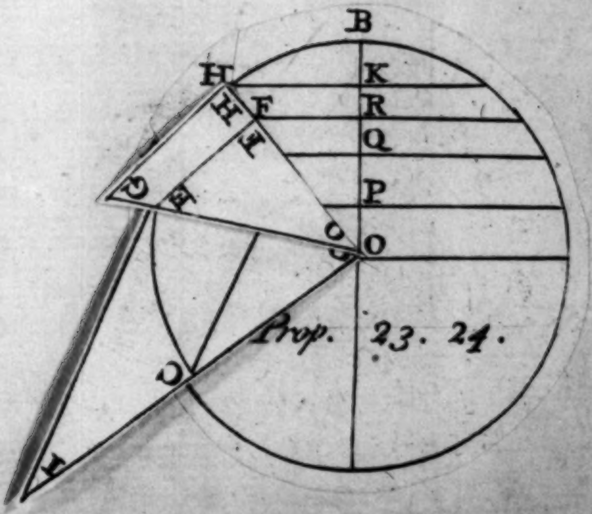
But

Page 60.



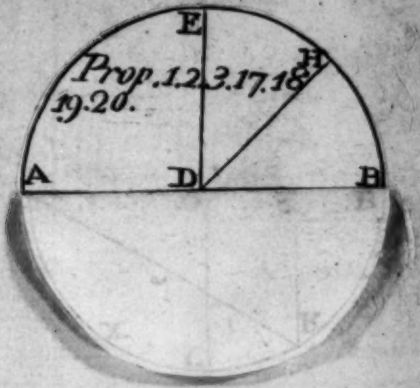


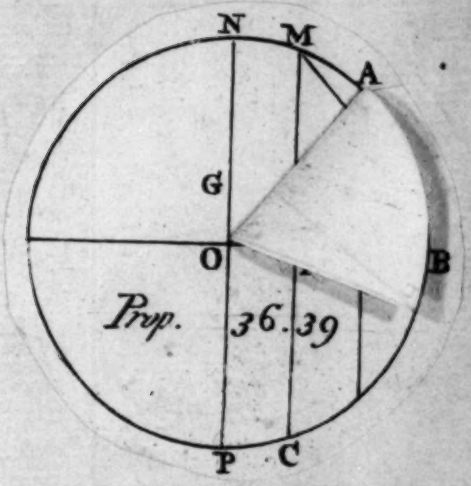
Prop. 36. 39

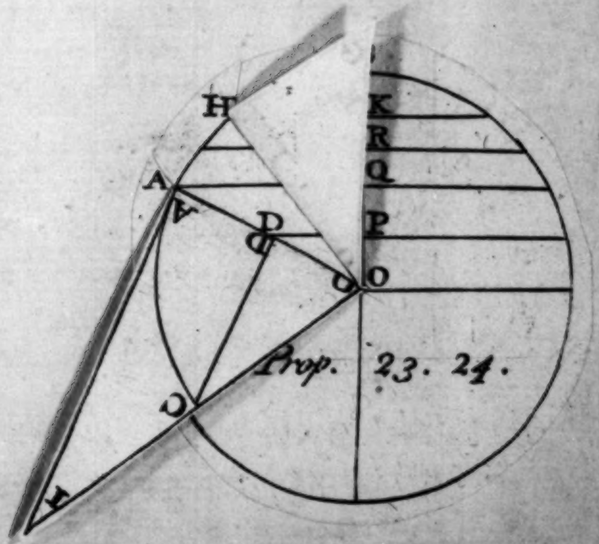


Prop. 23. 24.

Page 60.


E
H
Prop. 1.2.3.17.18
19.20.
A
D
B


N
M
A
G
O
B
Prop. 36.39
P
C


H
K
R
Q
A
A
D
D
P
O
O
Prop. 23. 24.
C
I

But if E V be a Quadrant, and at Right Angles to N X, then E, by 2d *Prop.* and its *Coroll.* is the Pole of N X; and ſo E N a Quadrant alſo, and the Angle E N V a Right one. Therefore, if the Sides of a Triangle (N E V, or its Equal) R S T, are equal to the Meaſures of the Angles of ſome other Triangle G H D, and the Meaſures of the Angles of the former, equal to the Sides of the latter; two Sides of ſuch a Triangle R S T, or G H D, muſt be Quadrants, and two Angles of each Right ones.

Therefore, if a Triangle R S T be conſtructed, whoſe Sides are equal to the Meaſures of the Angles of another Triangle G H D: The Meaſures of the Angles of the Triangle R S T, ſhall not be equal to the Sides of the Triangle G H D, unleſs in the one Caſe before-mentioned. Therefore the Meaſures of the Angles of the Triangle G H D, uſed as the Sides of a Triangle in the 11th Caſe, will not give us a Side of G H D, but the Meaſure of an Angle of the Triangle R S T, unleſs in the one afore-mentioned Caſe; *which was to be demonſtrated.*

But to find a Side G D of a Spherical Triangle G H D, whoſe Angles are all given, Produce M N, that Side of the ſupplemental Triangle, which is equal to the Supplement of the Meaſure of G H D, the Angle oppoſite to the Side ſought, and M X, either of the other Sides till they meet, as in E. And there, as hath been before ſhewn, the Sides E X, E N, of the Triangle E X N, are exactly equal to the Meaſures of the Angles H G D, G H D, of the Triangle G H D; and of the Angles E X N, E N X, of the Triangle E X N, the Meaſures are equal to G D, H D. But the Side X N is equal to the Supplement of the Meaſure of the Angle G D H. And of the Angle X E N, the Meaſure is equal to the Supplement of G H.

Therefore the SOLUTION is thus:

Change one of the Angles G D H, adjacent to the Side ſought into its Supplement; and then work with the Meaſures of the Angles, as tho' they were Sides, and the Reſult will be G D, the Side ſought.

The preceding Fault, as well as the Omiſſions hereafter-mention'd, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and many other Trigonometrical-Writers.

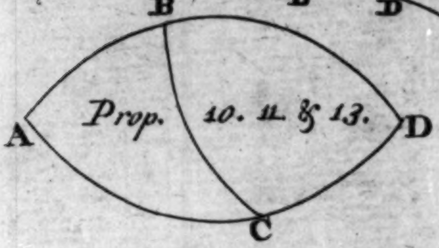
In the Solution of our 8th and 9th Caſes, they have told us, that the *Quæſita* are Ambiguous; which ſometimes, indeed, is true, but ſometimes alſo falſe: Therefore, as I conceive it, they ought to have laid down Rules, by help of which, we might diſcover when the *Quæſita* are Ambiguous, and when not.

This Over-ſight may be corrected by the following Directions: Wherein, becauſe every Sine correſponds to two Arches, to one leſs than a Quadrant, and to another, which is

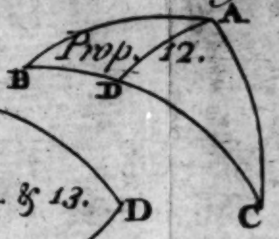
A
B
B
H
G
A
C

Spherical Trigonometry

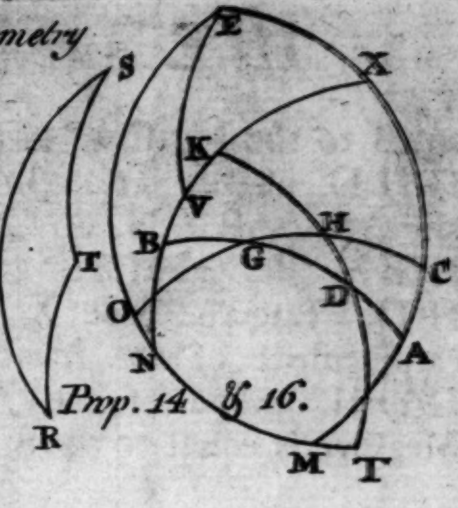
Page 63.



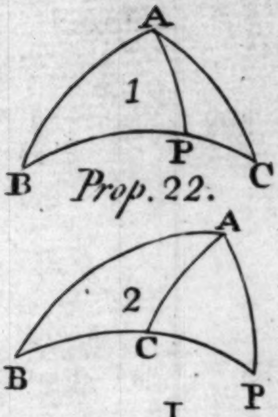
Prop. 10. 11. & 13.



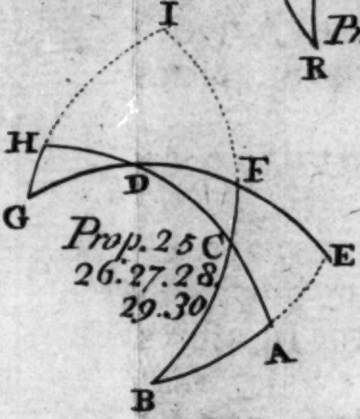
Prop. 12.



Prop. 14 & 16.



Prop. 22.



Prop. 25 26. 27. 28. 29. 30

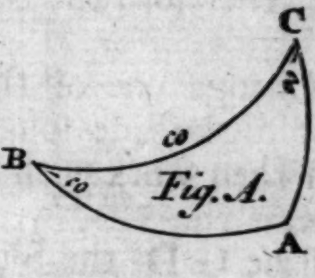


Fig. A.

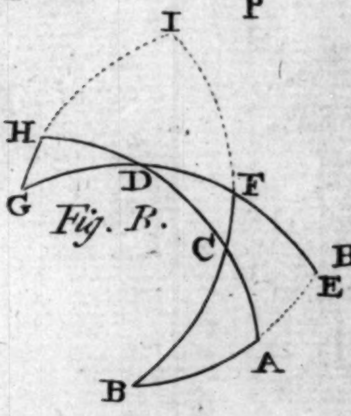
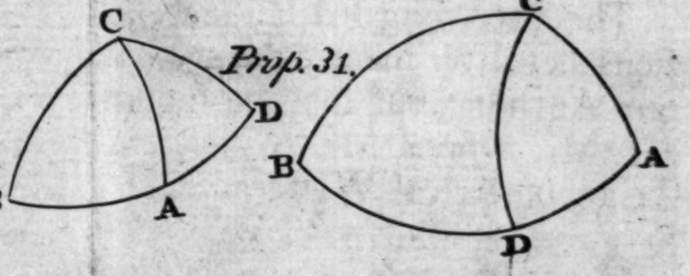
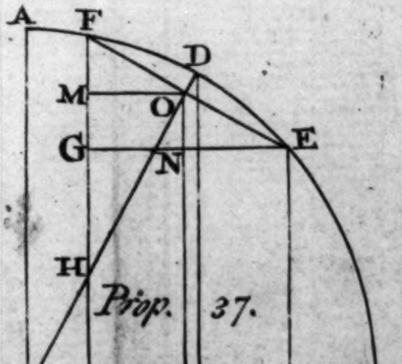


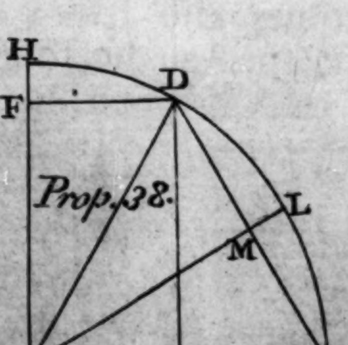
Fig. B.



Prop. 31.



Prop. 37.



Prop. 38.

the Supplement of the former to a Semicircle, (a true Diſtinction of which, of theſe are to be uſed, being neceſſary to be known, before a proper Solution can be given, to ſuch Problems as theſe are,) I ſhall beg leave, for Brevity-ſake, to call the leſſer Arch the Acute Value, and the greater the Obtuſe; whether the Sine be of an Angle, or a Side.

In the tenth Caſe, there are given two Angles B, D, *and* BC *a Side oppoſite to one of thoſe Angles* D, *to find* DC *the Side oppoſite to the other.*

TO the Acute Value of DC, and alſo to its Obtuſe one, add BC; and if each of theſe Sums are $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles; both the Values of DC may be admitted, and then is Ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, only one Value of DC can be true, *viz.* the $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute}\end{matrix}\right\}$ one; and then is not Ambiguous.

In the ninth Caſe, there are given two Sides BC, DC, *and one Angle* B, *oppoſite to* DC *one of thoſe Sides, to find* D *the Angle oppoſite to the other.*

TO the Acute Value of D, and alſo to its Obtuſe Value, add B; and if each of theſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, when the Sum of the Sides is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle; both the Values of D may be admitted, and conſequently, D is Ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, only one Value of D is true, *viz.* the $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute}\end{matrix}\right\}$ one; and then not Ambiguous.

Nor are we better uſed in the firſt Propoſition; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles, yet in each of thoſe Varieties, the *Quæſita* will be ſometimes Ambiguous, and ſometimes not.

In this firſt Propoſition, there are given two Angles B, D, *and* BC, *a Side oppoſite to* D, *one of them, to find* C *the third Angle.*

1. Let the Perpendicular fall within; that is, let the given Angles be of the ſame Species.

TO the Acute Value of DCA, and alſo to its Obtuſe one, add the Angle BCA; and if each of theſe Sums is leſs than two Right Angles, then either the Acute-Value of DCA, or its Obtuſe one added to BCA, gives a Value of BCD; which, therefore, is Ambiguous. And when only one of theſe

Sums is leſs than two Right Angles, the Acute-Value of DCA, added to BCA, gives the only Value of BCD; which then is not Ambiguous; tho' in both Varieties, the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of different Species.

WHEN the Obtuſe-Value of the Angle DCA is leſs than the Angle BCA, the Angle BCD may be had by ſubtracting either Value of DCA from BCA; and then BCD is Ambiguous. But when the Obtuſe-Value of DCA is not leſs than BCA, the Acute-Value of DCA, taken from BCA, gives the ſingle Value of BCD; which therefore is not Ambiguous; tho' in both Varieties, the Perpendicular fell without.

In the fifth Caſe, we lie under the ſame Misfortune, where there are given, as in the firſt, the Angles B, D, *and the Side* BC, *to find* BD, *the Side lying between thoſe given Angles.*

1. When the Perpendicular falls within; that is, when the given Angles are of the ſame Species.

TO the Acute-Value of DA, and ſo alſo to its Obtuſe one, add BA; and if each of theſe Sums is leſs than a Semicircle, then either the Acute-Value of DA, or its

Obtuſe one, added to BA, gives the Value of BD; which thence is Ambiguous. And when only one of theſe Sums is leſs than a Semicircle, the Acute-Value of DA, added to BA, gives the only Value of BD; which then is not Ambiguous; tho' in both Varieties, the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the Obtuſe-Value of DA is leſs than BA, BD will be had by ſubtracting either Value of DA from BA; and then BD is Ambiguous. But when the Obtuſe-Value of DA is not leſs than BA, the Acute-Value of DA, taken from BA, leaves the only Value of BD; which therefore is not Ambiguous; tho' in both Varieties, the Perpendicular fell without.

In the third, we have the ſame Omiſſion; where there are given two Sides BC, CD, *and* B, *an Angle oppoſite to* CD *one of them, to find the third Side* BD.

FIRST, we may obſerve, that the Species of DA is always known; for it is of $\left\{\begin{array}{l}\text{the ſame}\\ \text{a different}\end{array}\right\}$ Affection with the Angle B, when DC is $\left\{\begin{array}{l}\text{leſs}\\ \text{greater}\end{array}\right\}$ than a Quadrant. And,

If AD be leſs than AB, and alſo the Sum of AD and AB leſs than a Semicircle; then

AD,

A D, either added to, or ſubtracted from A B, will give the Value of B D, which, therefore, is Ambiguous.

But if A D be not leſs than A B, or if their Sum be not leſs than a Semicircle; then their Sum, in the former, and their Difference in the latter Variety, ſhall give one ſingle Value of B D, and then is not Ambiguous.

The ſeventh Caſe much reſembles the third; for there are given two Sides B C, C D, *and* B, *an Angle, oppoſite to* C D *one of them; to find the Angle* B C D, *lying between thoſe two Sides.*

AND here we may obſerve, that the Species of the Angle D C A is known; for it is of $\left\{\begin{array}{l}\text{the ſame}\\ \text{a different}\end{array}\right\}$ kind with the Angle B, when D C is $\left\{\begin{array}{l}\text{leſs}\\ \text{greater}\end{array}\right\}$ than a Quadrant. And,

If D C A be leſs than B C A, and the Sum of D C A and B C A, leſs than two Right-Angles; then, D C A either added to, or ſubtracted from B C A, will give the Angle B C D; which therefore is Ambiguous.

If D C A be not leſs than B C A, or the Sum of D C A and B C A, not leſs than two Right-Angles; then their Sum in the former, and their Difference in the latter Variety, ſhall give the Single-Value of B C D; which then is not Ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side D C, or the Angle D, as taught in the Remarks on the 9th and 10th Caſes, they will find the ſeveral Varieties in the 1ſt, 3d, 5th, and 7th, to be as here laid down in theſe eaſy Rules.

The Truth of theſe Rules may be eaſily deduced from the 10*th*, 13*th*, 18*th*, and 22*d* *Prop. of this*, and the 2*d*, 8*th*, and 13*th* *Examples*, following *Prop.* 30. *of this*.

In our third *Caſe* of Oblique-plain Triangles, our Author ſhould have added this.

If A B be leſs than B C, the Angle A is Ambiguous, otherwiſe not.

Sam

Sam. Cunn's Obſervations on the preceding Trigonometrical-Tract.

I find our Author, as well as Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and others of the Trigonometrical-Writers, is miſtaken in ſome of the Solutions.

That the common Solution of the 12th *Caſe* of Oblique-Spheric's is falſe, I have demonſtrated, and given a true one. *See Page* 59.

In the Solution of our 9th and 10th *Caſes*, by other Authors called the 1ſt and 2d, where are given and ſought oppoſite Parts, not only the afore-mention'd Authors, but all others, that I have met with, have told us, that the Solutions are Ambiguous; which Doctrine is, indeed, ſometimes true, but ſometimes falſe: For ſometimes the *Quæſitum* is doubtful, and ſometimes not; and when it is not doubtful, it is ſometimes greater than 90 Degrees, and ſometimes leſs: And ſure, I ſhall commit no Crime, if I affirm, that no Solution can be given without a juſt Diſtinction of theſe Varieties. For the Solution of theſe *Caſes*, ſee my Directions at *Pages* 62, 63.

In the Solution of our 3d and 7th *Caſes*, in other Authors reckon'd the 3d and 4th, where there are given two Sides and an Angle oppoſite to one of them, to find the 3d Side, or the Angle oppoſite to it ; all the Writers of Trigonometry, that I have met with, who have undertaken the Solutions of theſe two, as well as the two following *Caſes*, by letting fall a Perpendicular, which is undoubtedly the ſhorteſt and beſt Method for finding either of theſe *Quæſita*, have told us, that the $\left\{\begin{matrix}\text{Sum}\\ \text{Difference}\end{matrix}\right\}$ of the Vertical Angles, or Baſes, ſhall be the ſought Angle or Side, according as the Perpendicular falls $\left\{\begin{matrix}\text{within}\\ \text{without}\end{matrix}\right\}$ which cannot be known, unleſs the Species of that unknown Angle, which is oppoſite to a given Side, be firſt known.

Here they leave us firſt to calculate that unknown Angle, before we ſhall know whether we are to take the Sum, or the Difference of the Vertical-Angles or Baſes, for the ſought Angle or Baſe : And in the Calculation of that Angle, have left us in the dark as to its Species ; as appears by my Obſervations on the two preceding *Caſes*.

The Truth is the *Quæſitum* here, as well as in the two former *Caſes*, is ſometimes doubtful, and ſometimes not ; when doubtful, ſometimes each Anſwer is leſs than 90 Degrees, ſometimes each is greater ; but ſometimes one leſs, and the other greater, as in the two laſt mention'd *Caſes*. When it is not doubtful, the *Quæſitum* is ſometimes greater than 90 Degrees, and ſometimes leſs. All which Diſtinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, oppoſite to a given Side ; or which is the ſame thing, the falling of the Perpendicular, within or without. For which ſee my Directions at *Pages 66, 67.*

In the Solution of our 1ſt and 5th *Caſes*, called in other Authors, the 5th and 6th ; where there are given two Angles, and a Side oppoſite to one of them, to find the 3d Angle, or the Side oppoſite to it ; they have told us, that the $\left\{\begin{array}{l}\text{Sum}\\ \text{Difference}\end{array}\right\}$ of the Vertical Angles, or Baſes, according as the Perpendicular falls $\left\{\begin{array}{l}\text{within}\\ \text{without}\end{array}\right\}$ ſhall be the ſought Angle or Side ; and that it is known whether the Perpendicular falls within or without, by the Affection of the given Angles.

Here they ſeem to have ſpoken, as tho' the *Quæſitum* was always determin'd, and never Ambiguous ; for they have here determined, whether the Perpendicular falls within or without, and thereby whether they are to take the Sum, or the Difference of the Vertical-Angles or Baſes, for the ſought Angle or Side.

But, notwithſtanding theſe imaginary Determinations, I affirm, that the *Quæſitum* here, as in the two *Caſes* laſt mentioned, is ſometimes Ambiguous, and ſometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See my Solutions of theſe two *Caſes* in *Page* 64, 65.

The Determination of the 3d *Caſe* of Oblique-Plain Triangles. See in *Page* 66.

SAM. CUNN.

APPENDIX

By another Hand.

Containing the Elements of Aſtronomy, *by the Solution of* Spherical *Triangles upon the* Globe.

SPHERICAL Triangles are beſt repreſented by the Circles on, and thoſe *appendant* to the *Globes*. The *Globe* being the *Original*, from whence all *Spherical* Trigonometry proceeds, and upon which the Sides and Angles of all *Spherical* Triangles are moſt *naturally* repreſented, and moſt *Expeditiouſly* meaſured.

Moreover, as the whole Art of *Navigation* whether by plain, *Mercator's* or *Wright's* Sailing is but the Solution and Application of the Doctrine of *Plain* Triangles: ſo *Aſtronomy* is the bare Solution and Application of the Doctrine of *Spherical* Triangles.

For as the Poet ſay's.

Cuncta Trigonus habet, Satagit quæ docta Matheſis
Ille aperit Clauſum quicquid Olympus habet.

In *Trigonometry* encloſed lye,
All the Grand Secrets of *Aſtronomy*.

But before the Practiſe, it will be neceſſary to premiſe ſome general *Definitions*, and neceſſary *Explanations* of the *Aſtronomical* Circles and *Terms* hereafter uſed.

Firſt then.

1. A *Globe* is a round body, every part of whoſe Surface is equidiſtant from a point within it, called it's *Center*.

It may be conceived to be form'd by the revolution of a *Semicircle*, round it's *Diameter*. Every point in the Circumference of the generating *Semicircle* in it's circumvolution upon it's *Axis* or Diameter, marks out a Circle in the Surface of the *Globe*; whoſe Circles are greater or leſs as they are more remote from, or nearer to the Extreme points of the *Axis*, which are call'd it's Poles.

2. A *Great Circle* equally divides the *Globe* into two Hemiſpheres and has the ſame Center with it.

The moſt remarkable *Great Circles* are theſe the

Equator or *Equinoctial*,
Horizon.
Meridian.
Ecliptic.
Equinoctial Colure.
Solſtitial Colure.
Azimuth or *Vertical* Circles.

3. The *Equator*, by Sailors emphatically called the *Line*, is equidiſtant from the Poles of the World, which are alſo it's Poles, it Divides the Earth into N°. and S°. Latitude, *on* it is reckon'd the *Longitude* of places, and *from* it their Latitude is counted: The *Equinoctial* is it's correſponding Circle in the Heavens, cuts always the *Horizon* in the Eaſt or Weſt Points, and divides the Celeſtial *Globe* into N°. and S°. Declination: *on* it is reckoned *Right Aſcenſion*, and from it the *Declination* of the ☉, ☽, or Stars, it is graduated from 10. 20. to 360. and repreſented in Plate 5. by the right line ÆQ.

4. The *Horizon* is a *broad*, *wooden* Circle, dividing the *Globe* into two Hemiſpheres; the *upper* and *viſible*, which we ſee, from the lower and inviſible, which we ſee not, *on* it are the points of the Compaſs, Days of the Month and Signs, by this Circle is determined the *Riſing* and *Setting* of ☉, ☽ or Stars: *on* this we count their *Amplitude Azimuth* and ☉'s place every day in the Year: *from* it are counted the Almicanters or Parallels of Al-

titude; it's Poles are the *Zenith* or *Nadir*. H X O, in *Plate* 5, repreſents the *Horizon*.

5. The *Meridian* is a *broad braſs* Circle, paſſing thro' the Poles of the *Horizon*, and Poles of the World; croſſing both the *Equinoctial* and *Horizon* at right Angles, and divides the *Globe*, into the *Eaſtern* and *Weſtern* Hemiſpheres, it's Poles lie in the E. and W. Points of the *Horizon*, in this *Meridian* and the *Horizon* the *Globe* hangs and turns, it's graduated into four Nineties, *on* it is reckoned the *Latitude* of places, the *Declination* and *Meridian* Altitude of ☉, ☽ or Stars, and *from* it the *Longitude* of places and hour from Noon, 'tis repreſented in Plate 5. by the outmoſt Circle, Æ Z P H Q N S O.

6. *The Ecliptic* cuts the *Equinoctial* at Angles of 23°. 2′9; therefore it's Poles are diſtant from the Poles of the Equinoctial 23°. 29′. 'tis divided into 12 Signs each 30°. on either Side of the *Ecliptic*, about 8 Degrees off are two parallels which make the *Zodiac* or boundaries of the Planets. The *Ecliptic* divides the Celeſtial *Globe* into N°. and S°. Latitude of ☽ or Stars. *On* it is reckoned their *Longitude*, and *from* it their *Latitude* 'tis repreſented in Plate 5. by ♋ A ♑.

7. The *Equinoctial* Colure is that *Meridian*, which cutteth the *Equinoctial* at right Angles in thoſe two Points, where the *Ecliptic* croſſes the *Equinoctial* in ♈ and ♎, and ſo paſſes thro' the Poles of the World, in Plate 5. 'tis repreſented by P X S. 8.

8. The *Solſtitial* Colure is that *Meridian* which cutteth the *Equinoctial* Colure, the *Equinoctial* and the *Ecliptic*, all three at right Angles, and paſſes thro' the *Ecliptic*, where it touches both the *Tropic's* in ♋ and ♑. The outmoſt Circle in Plate 5. ÆZPHQNSO. repreſents it.

9. The *Azimuth* or *Vertical* Circles all paſs thro' the *Zenith* and *Nadir* at right Angles to the *Horizon*: as the *Meridians* paſs thro' the Poles of the World at right Angles, to the *Equinoctial*: that *Azimuth* Circle, which paſſes thro' the E. and W. points of the *Horizon* is called the *Prime Vertical* repreſented in Plate 5. by ZXN. The *Azimuths* are not drawn on *Globes*, but are repreſented by the *Quadrant* of *Altitude*, skrew'd on the *Zenith* which reaches to the *Horizon*, and is divided into 90°. for numbring the Degrees of Altitude, when the ☉, ☽ or Stars are not on the *Meridian*.

10. A *Smaller Circle* divides the *Globe* into two unequal Parts. The chief are the Parallels of.

Declination.
Altitude.
Latitude.

11. The *Parallels* of *Declination* are all thoſe *leſs* Circles on the *Globe*, which are drawn parallel to the *Equinoctial*. Four of which are moſt noted. The two *Tropic's* of ♋ and

♑

♑ at 23°. 29′ from the *Equinoctial* both ways, and are the bounds of the ☉'s greateſt *Declination* N°. and S°. markt in plate 5. ♋ ♋, ♑ ♑. Alſo the 2 *Polar* Circles, as far diſtant from the Poles of the World, as the *Tropics* are from the *Equinoctial*, that next the N°. Pole called the *Artic.* and that next the S°. Pole called the *Antartic* Circle, both which are bounds of the 2. *frigid* Zones, as the *Tropics* are of the *Torrid* Zone, betwixt the *Torrid* and the *Frigid* lie the two Temperate.

12. The *Parallels* of *Altitude* are all thoſe *Leſs* Circles, which are parallel to the *Horizon*, repreſented by the ſeveral Degrees of the *Quadrant* of Altitude moved about parallel, to the *Horizon* called often *Almicanters.*

13. The *Parallels* of *Latitude* on the *Terreſtrial* Globe, are the ſame as *Parallels* of *Declination* on the *Celeſtial*; both being parallel to the *Equinoctial* : but *Parallels* of *Latitude* on the *Celeſtial* Globe, are thoſe *Smaller* Circles parallel to the *Ecliptic*, and repreſented by the degrees of the *Quadrant* of *Altitude*, skrew'd on the Poles of the *Ecliptic*, and moved parallel to the *Ecliptic*. The ſaid *Quadrant* repreſenting then the Circles of Longitude of the Stars.

14. The Principal Points are the Poles of the
{ *World*, or of the *Equinoctial.*
{ *Horizon.*
{ *Meridian.*
{ *Ecliptic.*

{*Equinoctial Colure.*
{*Solſtitial Colure.*
{*Prime Vertical.*

15. The *Poles* of the *World*, or of the *Equinoctial* being the ſame, are the two points on which the *Globe* hangs and turns within the *Brazen Meridian* and *Wooden Horizon*: from one of which to the other, there paſſes a ſtrait line thro' the Center of the *Globe* called the *Axis* of the World, in theſe two all *Meridians* meet: in *Plate* 5, they are markt P. and S.

16. The *Poles* of the *Horizon* are called the *Zenith* and *Nadir*, markt in Plate 5, Z. and N. the *Zenith* is that point in the Heavens directly over our Heads: and the *Nadir* is the contrary, directly under our feet, in theſe two do all Azimuth or Vertical Circles meet.

17. The *Poles* of the *Meridian* are the points of Eaſt and Weſt, where the Equinoctial cuts the Horizon and are repreſented in Plate 5. by the point X. the other being inviſible, being diametrically oppoſite.

18. The *Poles* of the *Ecliptic* are 23°. 29′ diſtant from the Poles of the World, in which two points the Circles of Longitude of the Stars do croſs each other, which Circles are repreſented by the Quadrant of Altitude skrew'd over the Pole of the Ecliptic.

19. The *Poles* of the *Equinoctial Colure* are Æ. Q. where the Solſtitial Colure croſſes the Equinoctial. And,

20. The *Poles* of the *Solſtitial Colure* is X. where the *Equinoctial Colure* croſſes the *Equinoctial* and *Ecliptic* in ♈ and ♎, which four poles are Cardinal points, and divide the Year into 4 Quarters, the

Spring when ☉ appears in ♈
Summer ♋
Autumn ♎
Winter ♑

21. The two *Solſtitial points* are ♋ and ♑ where the Ecliptic touches the 2 Tropics, which the ☉ appears to do, when the days are longeſt or ſhorteſt, and cauſes them to ſtand ſtill a while, without a Senſible lengthning or ſhortning, for the ſpace of 10 or 12 days.

22. The *Poles* of the *Prime Vertical* are the N°. and S°. points of the Horizon, thro' which the *Brazen Meridian* paſſes.

23. The *Aſtronomical* Terms following, which need Explanation are.

Declination
Amplitude
Altitude
Azimuth
Latitude
Longitude.

☉'s Poſition
Right Aſcenſion
Oblique Aſcenſion
and
Aſcenſional Difference

24. *Declination* is an Arch of the *Meridian*, between the *Equinoctial* and the Star brought to the *Meridian*, and is either N°. or S°.

25. *Amplitude* is an Arch of the *Horizon* between the Eaſt or Weſt-points thereof, and the Star brought to the *Horizon*, and is either *Ortive* when the Star riſes, or Occidual when the Star ſets.

26. *Altitude* is an Arch of the *Quadrant of Altitude*, between the *Horizon* and the *Star*, brought to the *Quadrants* Edge, when ſkrew'd on the *Zenith*.

27. *Azimuth* is an Arch of the *Horizon* between the *Meridian*, and the Stars *Vertical*, repreſented by the *Quadrant* on the *Zenith*, when laid over the Star.

☞ By Star, in theſe 4 laſt Definitions underſtand ☉, ☽, Star, or any point in the Heavens.

28. *Latitude* is either *Celeſtial* or *Terreſtrial*. *Celeſtial* is the neareſt diſtance of a Star from the *Ecliptic*, meaſur'd by a Circle of Longitude repreſented by the *Quadrant of Altitude*, laid over the Pole of the Ecliptic and the Star, and is either N°. or S°. as the Star is Situate.

Latitude on the *Terreſtrial* Globe, is the neareſt diſtance of a place from the *Equator*, meaſured by an Arch of the *Brazen Meridian*, between the Equator and place, when brought to the *Meridian*, which *Latitude* is always equal to the Elevation of the *Pole* above the *Horizon*: becauſe it's Complement added to each, will make either of them Exact Quadrants.

drants. 'Tis alſo either N°. or S°. according to the Situation of the Place.

29. *Longitude* is either *Celeſtial* or *Terreſtrial*. *Celeſtial*, as the *Longitude* of the ☉ is it's place in the *Ecliptic*. And *Longitude* of a Star, is an Arch of the *Ecliptic*, between ♈ and that *Circle* of *Longitude* which paſſes thro' that Star; repreſented by the Edge of the *Quadrant*, skrew'd over the Pole of the *Ecliptic*, and laid over the Star.

Longitude of a place on the Terreſtrial Globe is an Arch of the *Equator* between the firſt *Meridian*, and the Meridian that paſſes thro' the given place, repreſented by the *Brazen Meridian*, when the place is brought to it.

30. The Angle of the ☉'s *Poſition* is form'd by the Concurrence of two great Circles at the point of the ☉'s place in any Spherical Triangle.

31. *Right Aſcenſion*, is an Arch of the *Equinoctial*, reckoning towards the Eaſt from ♈. to that Meridian, which paſſes thro' the ☉ or Star.

32. *Oblique Aſcenſion* is an Arch of the *Equinoctial*, reckoning towards the Eaſt, from ♈. to that point, which riſes with the ☉ or Star in an Oblique Sphere.

33. *Aſcenſional* Difference is the Difference in Degrees, between the *Right* and *Oblique*

Aſcenſion aforeſaid, which reduced into time, is the ſpace between ſix a Clock and ⊙ Riſe.

34. To reduce Degrees into Time, ſeeing 360° are Equal to 24 Hours.

For every		allow	H.
	15°.		1
	1		.. 4′
	.15′		1
	1		.. 4″

35. Having Sun-riſe, to find Sun-ſet, Length of Day, Length of Night, all the Year round.

Seeing we begin our Count at Mid-day, and Mid-night, the time of ⊙ Riſe, is conſequently one half of the Night, and time of ⊙ Set, one half of the Day: and the hours of both added together, muſt make up 12. wherefore ſubtract the hours of ⊙ Riſe from 12, the remains is the time of ⊙ Set, and double the time of ⊙ Set, gives the length of the Day, and double the time of ⊙ Riſe, gives the length of the Night all the Year round.

To Project the *Sphere* for the Latitude 51°. 30′ N°. Geometrically.

QUarter the Circle, which repreſents the *Solſtitial-Colure* or *Meridian*, with Z N. the *Prime-Vertical*, and H O the *Horizon*: then ſet the *Chord* of the Latitude 51. 30. from H to P, the *Pole* from Z, the *Zenith* to Æ, the *Equinoctial* from O to S, and from

N to Q, and draw Æ. Q the *Equinoctial*, and P S the *Equinoctial-Colure*, from Æ and Q on both ſides, ſet off the *Chord* of 23° $\frac{1}{2}$ for the *Tropic's* ♋ and ♑, and draw ♋ X ♑ the *Ecliptic*; lay a Rule from Æ to ♋, which cuts P S in L, then thro' the 3 points ♋ L ♋, draw the *Tropic* of ♋, which ſuppoſe the apparent Courſe of the *Sun* for that Day, which cuts the *Horizon* at ☉, thro' the 3 points P. ☉. S. draw the *Meridian* P ☉ S, which forms the 1ſt. and 3d. *Triangles* ſhaded, and at *6* a Clock, the ☉ appears at L, thro' Z. L. N. 3 Points, draw the Vertical Z L E N, which forms the 4th ſhaded *Triangle*, X L E. Again, ſuppoſe the ☉ at M, thro' the 3 points Z. M. N. draw the *Azimuth* Z M N, and draw thro' P. M. S. the *Meridian* P M S, thus is formed both the 2d and 5th ſhaded *Triangles* X y z, and P Z M.

Caſe 1.

IN the Right-Angled Spherical-Triangle H ☉ P, Fig. 1. Plate 5. Right-Angled at H are Given.

{P H. the Latitude 51. 30.
{P ☉. the ☉'s Diſtance from the Pole *69*. 30.

Required {H ☉. the Complement of ☉'s Amplitude.
{H P ☉. the Hour of ☉'s Riſing, accounted from Midnight.
{P ☉ H. the Angle of ☉'s Poſition, which the *Horizon* makes with the Meridian.

To repreſent this *Triangle* on the *Globe*.

Elevate the Pole to the given *Latitude*, and turn the *Globe* about, till 69° ½ of the *Equinoctial-Colure* counted from the *Pole*, be at the *Horizon*, ſo will the *Triangle* be perfectly deſcribed by an *Arch* of the Braſs *Meridian*, an *Arch* of the *Horizon*, and an *Arch* of the *Colure*.

The Complement of the ☉'s *Amplitude*, H ☉. is numbred on the *Horizon*, from the *Colure* to the N°. point of the *Horizon*, and in this caſe is 57°. 34′.

For the Angle H P ☉, count the degrees of the *Equinoctial*, which are contained between the *Colure* and the N°. ſide of the Braſs *Meridian*, which degrees reduced to time, *i. e.* 62°. 45′. are equal to 4 *h.* 11′ the hour from Midnight, the time of ☉ Riſe, and ½ the length of the Night, which ſubduct from 12, reſts 7 *h.* 49′ the time of ☉ Set, and half the length of the Day.

For the Angle P ☉ H, number on the *Colure*, and alſo on the *Horizon*, the Sides that include the Angle 90°. from the Angular Point, and the diſtance between thoſe two Points, where the reckoning ends, meaſured by the *Quadrant* of *Altitude*, or by Compaſſes, contains 56°. 39′. the Angle of the ☉'s Poſition.

Caſe 2.

IN the Right-Angled Spherical *Triangle*, X y z. Right-Angled at y, Fig. 2. Plate 5.

Given { X z. ⊙'s Diſtance from ♈ 72°.
y X z. ⊙'s greateſt *Declination* 23° 29'.

Required { z y. ⊙'s preſent *Declination.*
X y. ⊙'s *Right-Aſcenſion.*
y z X. ⊙'s Poſition, or the Angle that the *Meridian* makes with the *Ecliptic.*

To repreſent this *Triangle* on the *Globe.*

Bring the ⊙'s Place, which is ♊ 12°. to the Brazen *Meridian*, and fix the *Globe* in that Poſition, and the Triangle will be repreſented by an *Arch* of the *Equinoctial*, an *Arch* of the *Ecliptic*, and an *Arch* of the Brazen *Meridian.*

The ⊙'s preſent *Declination* X y. 22°. 16. is meaſured on the Brazen *Meridian*, from the *Equinoctial*, to the ⊙'s Place, and is N°. becauſe betwixt the *Equinoctial* and N°. Pole.

The ⊙'s *Right-Aſcenſion* X y, is meaſured on the *Equinoctial* 70° ½ from ♈, to the Braſs *Meridian.*

And for the Meaſure of the Angle y z X number 90°. on the *Meridian* from the ⊙'s place, and 90° on the *Ecliptic* from the ⊙'s place, ſo ſhall the diſtance between theſe two Points, meaſured by the *Quadrant* of *Altitude*, contain for the Angle y z X, 82°. 20′.

Caſe 3.

IN the Right-Angled *Spherical-Triangle* A ⊙ B, Right-Angled at B, Fig. 3. Plate 5.

Given { ⊙ A B. the Complement of Latitude 38°. 30′.
B ⊙. the ⊙'s Declination 20. 30. N°.

Required { A ⊙. ⊙'s Amplitude.
A B. the Aſcenſional-Difference.
A ⊙ B. ⊙'s Poſition at Riſing.

To repreſent this *Triangle* on the *Globe.*

Depreſs the *Equinoctial* below the N°. point of the *Horizon* 38°. 30′ the Complement of the Latitude, then number 20° ½ the ⊙'s Declination on the *Equinoctial-Colure*, and bring that to the Eaſt-ſide of the *Horizon*, ſo ſhall the *Triangle* be formed, and appear below the *Horizon* by A ⊙, an *Arch* of the *Horizon.* A B, an *Arch* of the *Equinoctial.* And B ⊙, an *Arch* of the *Colure.*

A ⊙ the ⊙'s *Amplitude* from the *Eaſt*, is numbred on the *Horizon* to the *Colure*, and is 32°. 26′.

A B the *Aſcenſional-Difference*, is numbred on the *Equinoctial*, from the Eaſt-point of the *Horizon* to the *Colure*, and is 27°. 15′. which reduced into time, is 1 *h*. 49′ before 6. for ⊙ Riſe *i. e.* 4. *h.* 11′.

And the Angle of the ⊙'s Poſition A ⊙ B, is found by continuing the Sides ⊙ A and ⊙ B to Quadrants, and then meaſuring betwixt thoſe two Points, as in the firſt and 2d Caſes.

Caſe 4.

IN the Right-Angled Spherical-Triangle A E L, Right-Angled at E, Fig. 4. Plate 5.

Given { AL the ⊙'s Declination = 20°. 30′ N°.
LAE the Latitude = 51. 30 N°.

Required { L E the ⊙'s *Altitude* at Six.
A E the ⊙'s *Azimuth* from Eaſt.
A L E the ⊙'s Poſition or Angle, which the *Colure* makes with the Vertical at Six.

To repreſent this *Triangle* on the *Globe*.

Elevate the *Pole* to the given *Latitude*, fix the *Quadrant* of *Altitude* in the *Zenith*, and number the ⊙'s *Declination* 20° ½ on the *Equinoctial-Colure*, the *Colure* being brought to the Eaſt-point of the *Horizon*, bring the graduated *Edge* of the *Quadrant*, over the ⊙

⊙'s *Declination* on the *Colure*, ſo will the Triangle be form'd by,

A L an *Arch* of the *Colure* = *Declination*.
A E an *Arch* of the *Horizon* = *Azimuth*, from the Eaſt.
L E an *Arch* of the *Quadrant* = ⊙'s *Altitude*.

☞ A E, ⊙'s *Azimuth* from E. is 13°. 6′.
L E, ⊙'s *Altitude* at 6. is 15. 54.
and the Angle of the ⊙'s Poſition A L E, is found by continuing the Sides L A and L E to Quadrants, and then meaſuring betwixt thoſe two Points, as in the firſt and ſecond Caſes.

Of *Oblique-Triangled Spherical-Triangles.*

Caſe 1.

IN the *Oblique-Angled Spherical-Triangle* P Z M, Fig. 5. Plate 5. *Oblique-Angled* at Z.

Given:
Z P the Complement of the Latitude = 38°. 30′.
M P the ⊙'s Diſtance from the Pole = *69*. 30.
M Z the Complement ⊙'s Altitude = 47. 20.

Required:
P Z M the ⊙'s *Azimuth* from the N°.
Z P M the hour of the Day from Noon.
Z M P the ⊙'s Angle of Poſition.

To repreſent this *Triangle* on the *Globe.*

Elevate the *Pole* to the *Latitude*, then count Z P 38°. 30′. the Complement of the *Latitude* from the *Pole*, on the S°. Side of the brazen *Meridian*, and where the Reckoning ends, skrew the *Quadrant* of *Altitude i. e.* on the *Zenith*, which is markt 51°. 30′. on the braſs *Meridian*, this done, count the Side M P 69° $\frac{1}{2}$. upon the *Equinoctial-Colure* from the *Pole*, and the Side M Z = 47°. 20′. on the Quadrant of *Altitude*, from the *Zenith* downwards, and ſo move the *Globe* and *Quadrant* of *Altitude* together, till the Numbers counted on the *Colure* and *Quadrant* of *Altitude*, meet in one Point, ſo will the *Triangle* be exactly delineated upon the *Globe.*

Now to find the ſeveral *Angles.*

For the *Angle* Z, count the Number of Degrees, of the *Horizon*, which are contain'd between the N°. Point of the *Meridian*, and the *Quadrant* of *Altitude* 115°. 35′. the ☉'s *Azimuth* from the N°.

For the Angle Z P M, the degrees of the *Equinoctial*, contain'd between the S°. ſide of the Braſs *Meridian*, and the *Colure* 45° reduced into time, is 3 Hours from Noon, *i. e.* 9 in the Morning, or 3 Afternoon.

For the Angle Z M P, the Angle of Poſition, reckon 90° from the point M, on the

Circles M Z and M P, the diſtance betwixt thoſe Points meaſured by the Quadrant of Altitude, applied to both, give 36°. 52′. the Angle ſought made by the *Meridian* and *Vertical*, concurring in M.

Alſo by having any 3 Parts of any Oblique-Triangle given, the other Three may be found, which afford no leſs Variety, than 60 Caſes in an Oblique, and 30 in a Right.

As will appear, if we call the Latitude 1. ☉'s Declination 2. Azimuth 3. Altitude 4. Poſition 5. Hour 6. Then may each of theſe be found from 10 different Data's.

	1	2	3	4	5	6
may be found from the Ten different *Data's*, directly underneath.						
1	2.3.4	1.5.6	1.5.6	1.5.6	2.3.4	2.3.4
2	2.3.5	1.4.6	1.4.6	2.3.5	1.4.6	2.3.5
3	2.3.6	1.4.5	1.4.5	2.3.6	2.3.6	1.4.5
4	2.4.5	1.3.6	2.4.5	1.3.6	1.3.6	2.4.5
5	2.4.6	1.3.5	2.4.6	1.3.5	2.4.6	1.3.5
6	2.5.6	1.3.4	2.5.6	2.5.6	1.3.4	1.3.4
7	3.4.5	3.4.5	1.2.6	1.2.6	1.2.6	3.4.5
8	3.4.6	3.4.6	1.2.5	1.2.5	3.4.6	1.2.5
9	3.5.6	3.5.6	1.2.4	3.5.6	1.2.4	1.2.4
10	4.5.6	4.5.6	4.5.6	1.2.3	1.2.3	1.2.3

Thus may we ſerve the five Parts of a Right Angled, Spherical Triangle.

Moreover, in the ſame Triangle, call the Side Z P, the diſtance of the Pole of the *Equinoctial*, from the Pole of the *Ecliptick*, then P M will be the Co-declination of a *Star* at M. Z M will be the Complement of the N°. *Latitude* of that *Star*, the Angle

MPZ, will be the Complement of the *Stars* Right-Aſcenſion; the Angle **PZM**, the *Stars-Longitude* from ♋, and **PMZ** the Angle of the *Stars* Poſition.

The ſame Triangle may be applied to Sundry other uſes in *Geography*, *Navigation*, and *Dialling*, for which we have no room in this ſmall *Appendix*.

ERRATA.

	For	Read
P. 24. li. 14.	AB : BC.	AB : AC.
p. 72. li. 8.	page *66*.	p. *68*.

Alſo omitted in the *London* Edition.

p. 10. lin. 14. after *Sine* of, Read 2° to the Difference of the *Sines* of 13 and 17°. and ſo is the *Sine* of p. 33. lin. 15. after *Prop.* 2. Read and **XB** = Quadrant.

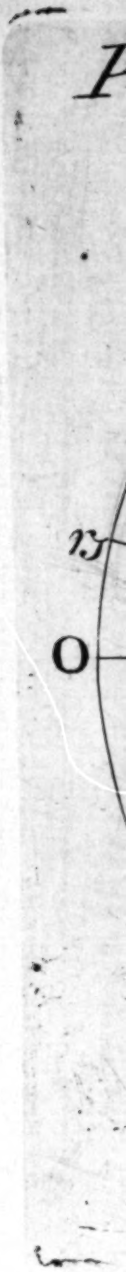
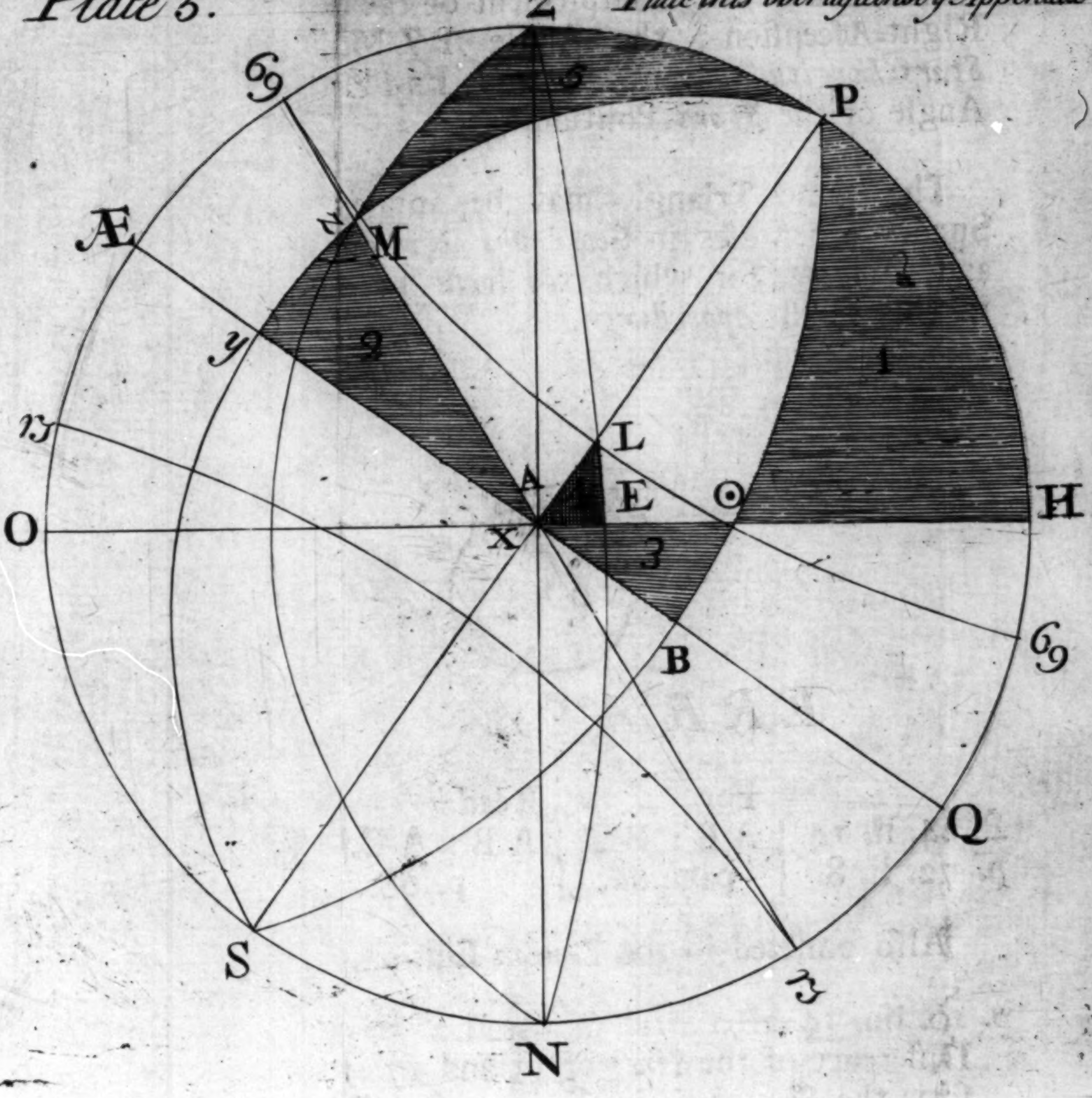

P
n
O

Plate 5.

Place this over against yᵉ Appendix



A Short

TREATISE

OF THE

NATURE and ARITHMETICK

OF

LOGARITHMS.

THE

PREFACE.

T*H E Mathematicks formerly received conſiderable Advantages; firſt, by the Introduction of the* Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it ſince reaped at leaſt as much from the Invention of Logarithms, as from both the other two. The Uſe of theſe, every one knows, is of the greateſt Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almoſt infinite, and ſuch as are otherwiſe impracticable, are managed with Eaſe and Expedition. By their Aſſiſtance the Mariner ſteers his Veſſel, the Geometrician inveſtigates the Nature of the higher Curves, the Aſtronomer determines the Places of the Stars, the Philoſopher accounts for other Phænomena of Nature; and laſtly, the Uſurer computes the Intereſt of his Money.*

The Subject of the following Treatiſe has been cultivated by Mathematicians of the firſt Rank; ſome of whom taking in the whole Doctrine, have indeed wrote learnedly, but ſcarcely intelligible to any but Maſters. Others, again, accommodating themſelves to the Apprehenſion of Novices, have ſelected out ſome of the moſt eaſy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Deſign therefore in the following Tract, is to ſupply what ſeemed ſtill wanting, viz. *to diſcover and explain the Doctrine of Logarithms, to thoſe who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the firſt that conſtructed and publiſhed a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciouſly received by all Mathematicians, who were immediately ſenſible of the extreme Uſefulneſs thereof. And tho' it is uſual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is univerſally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The ſame Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr.* Henry Briggs, *Profeſſor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the completing thereof: But the Lord* Neper *dying, the whole Buſineſs remaining was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compaſs'd a Logarithmic Canon, agreeable to that new Form for the* *firſt twenty Chiliads of Numbers,* (*or from* 1 *to* 20000) *and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publiſh'd at* London *in the Year* 1624.

Adrian Vlacq *publiſhed again this Canon at* Goudæ *in* Holland *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Brigg's *Preſcriptions; but theſe Tables are not ſo uſeful as* Brigg's, *becauſe the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *alſo has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has ſubjoined to them the Natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themſelves are called Natural. Theſe Tables, together with their Conſtruction and Uſe, were publiſh'd after* Brigg's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called* Trigonometria Britannica.

Since then, there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

The beſt Diſpoſition of theſe Tables, in my Opinion, is that, firſt thought of by Nathaniel Roe, *of* Suffolk; *and with ſome Alterations for* *the better, followed by* Sherwin *in his Mathematical-Tables publiſh'd at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000 *conſiſting of* 7 *Places of Figures. To which are ſubjoined the Differences and proportional Parts, by Means of which may be found eaſily the Logarithms of Numbers to* 10000000, *obſerving at the ſame Time that theſe Logarithms conſiſt only of* 7 *Places of Figures. Here are alſo the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Vſe in practical Mathematicks.*

OF THE NATURE and ARITHMETICK OF *LOGARITHMS.*

CHAP. I.

Of the ORIGIN *and* NATURE *of Logarithms.*

AS in Geometry, the Magnitudes of Lines are often defined by Numbers; ſo likewiſe on the other hand, it is ſometimes expedient to expound Numbers by Lines, *viz.* by aſſuming ſome Line which may repreſent Unity, and the Double thereof; the Number 2, the Triple 3, the one half, the Fraction $\frac{1}{2}$, and ſo on. And thus the Geneſis and Properties of ſome certain Numbers are better conceived, and more clearly conſidered, than can be done by abſtract Numbers.

* *Fig.* 1. Hence, if any Line a * be drawn into itſelf, the Quantity a^2 produced thereby, is not to be taken as one of two Dimenſions, or as a Geometrical Square, whoſe Side is the Line a, but as a Line that is a third Proportional to ſome Line taken for Unity and the Line a. So likewiſe, if a^2 be multiplied by a, the Product a^3, will not be a Quantity of three Dimenſions, or a Geometrical-Cube, but a Line that is the fourth Term in a Geometrical-Progreſſion, whoſe firſt Term is 1, and ſecond a; for the Terms $1, a, a^2, a^3, a^4, a^5, a^6, a^7$, *&c.* are in the continual Ratio of 1 to a. And the Indices affixed to the Terms, ſhew the Place or Diſtance that every Term is from Unity. For Example, a^5 is in the fifth Place from Unity, a^6 is the ſixth, or ſix times more Diſtant from Unity, than a, or a^1, which immediately follows Unity.

If between the Terms 1 and a, there be put a mean Proportional which is $\surd \bar{a}$, the Index of this will be $\frac{1}{2}$, for it's Diſtance from Unity will be one half of the Diſtance of a from Unity; and ſo $a^{\frac{1}{2}}$ may be written $\surd \bar{a}$. And if a mean Proportional be put between a and a^2, the Index thereof will be $1\frac{1}{2}$ or $\frac{3}{2}$, for it's Diſtance will be ſeſquialteral of the Diſtance of a from Unity.

If there be two mean Proportionals put between 1 and a; the firſt of them is the Cube Root of a, whoſe Index muſt be $\frac{1}{3}$, for that Term is diſtant from Unity only by a third Part of the Diſtance of a from Unity;

and ſo the Cube-Root muſt be expreſſed by $a^{\frac{1}{3}}$. Hence, the Index of Unity is 0. for Unity is not diſtant from it ſelf.

The ſame Series of Quantities, Geometrically Proportional, may be both ways continued, as well deſcending towards the Left Hand, as aſcending towards the Right; for the Terms $\frac{1}{a^5}$ $\frac{1}{a^4}$ $\frac{1}{a^3}$ $\frac{1}{a^2}$ $\frac{1}{a}$ 1, a, a^2, a^3, a^4, a^5, &c. are all in the ſame Geometrical Progreſſion. And ſince the Diſtance of a from Unity is towards the Right Hand, and poſitive or $+1$, the Diſtance equal to that on the contrary Side, *viz.* the Diſtance of of the Term $\frac{1}{a}$ will be Negative or -1, which ſhall be the Index of the Term $\frac{1}{a}$ for which may be written a^{-1}. So likewiſe in the Term a^{-2}. The Index -2 ſhews that that Term ſtands in the ſecond Place from Unity towards the Left Hand, and the Terms a^{-2} and $\frac{1}{a^2}$ are of the ſame Value. Alſo a^{-3} is the ſame as $\frac{1}{a^3}$ For theſe negative Indices ſhew that the Terms belonging to them, go from Unity the contrary way to that, by which the Terms whoſe Indices are poſitive do. Theſe Things premiſed.

If on the Line A N, Fig. 2. both ways indefinitely extended, be taken, AC, CE, EG, GI, I L, on the right Hand. And alſo A r, r π, *&c.* on the left, all equal to one another. And if at the Points π, r, A, C, E, G, I, L, be erected to the Right Line A N, the Perpendiculars π ≷, r Δ, A B, C D, E F, G H, I K, L M, which let be continually proportional, and repreſent Numbers, whereof A B is Unity. The Lines A C, A E, A G, A I, A L—A r, —A π, reſpectively expreſs the Diſtances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical-Proportionals, according as it is diſtant from Unity. So ſince A G is triple of the Right Line A C, the Number G H ſhall be in the third Place from Unity, if C D be in the firſt: So likewiſe ſhall L M be in the fifth Place, ſince A L = 5 A C. If the Extremities of the Proportionals ≷, Δ, B, D, F, H, K, M, be joined by Right Lines, the Figure ≷ π L M will become a Polygon conſiſting of more or leſs Sides, according as there are more or leſs Terms in the Progreſſion.

If the Parts A C, C E, E G, G I, I L, be biſected in the Points *c*, *e*, *g*, *i*, *l*, and there be again raiſed the Perpendiculars *c d*, *e f*, *g h*, *i k*, *l m*, which are mean Proportionals between A B, C D; C D, E F; E F, G H; G H, I K; I K, L M; then there will ariſe a new Series of Proportionals, whoſe Terms beginning from that, which immediately follows Unity, are double of thoſe in the firſt Series, and the Difference of the Terms are become leſs, and approach nearer to a Ratio of Equality than before. Likewiſe in this new Series, the Right Lines A L, A C, expreſs the Diſtances of the Terms L M, C D, from Unity, *viz.* Since A L is ten times greater than A *c*, L M ſhall be the tenth Term of the Series from Unity: And becauſe A *e* is three times greater than A *c*, *e f* will be the third Term of the Series, if *c d* be the firſt; and there ſhall be two mean Proportionals between A B and *e f*, and between A B, and L M, there will be nine mean Proportionals.

And if the Extremities of the Lines B *d* D *f* F *h* H, &c. be joined by Right Lines, there will be a new Polygon made conſiſting of more, but ſhorter Sides than the laſt.

If again, the Diſtances A *c*, *c* C, C *e*, *e* E, &c. be ſuppoſed to be biſected, and mean Proportionals between every two of the Terms, be conceived to be put at thoſe middle Diſtances; then there will ariſe another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Differences of the Terms will be leſs; and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be leſs, becauſe of the Diminution of the Diſtances of the Terms from each other.

Now in this new Series, the Diſtances A L, A C, &c. will determine the Orders or Places of the Terms; *viz.* if A L be five times greater than A C, and C D be the fourth Term of the Series from Unity, then L M will be the twentieth Term from Unity.

If in this Manner, mean Proportionals be continually placed between every two Terms, the Number of Terms at laſt will be made ſo great, as alſo the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon ſo leſſened, as to become leſs than any given Right Line; and conſequently the Polygon will be changed into a Curv-lin'd Figure; for any Curv-lin'd Figure may be conceived as a Polygon, whoſe Sides are infinitely ſmall and infinite in Number.

A Curve deſcribed after this Manner, is called *Logarithmical*; in which, if Numbers be repreſented by Right Lines ſtanding at Right Angles to the Axis A N, the Portion of the Axis intercepted between any Number and Unity, ſhews the Place or Order that that Number obtains in the Series of Geometrical Proportionals, diſtant from each other by equal Intervals. For Example, if A L be five times greater than A C, and there are a thouſand Terms in continual Proportion from Unity to L M; then will there be two Hundred Terms of the ſame Series from Unity to C D, or C D ſhall be the two hundredth Term of the Series from Unity; and let the Number of Terms from A B to L M be ſuppoſed what it will; then the Number of Terms from A B to C D, will be one fifth of that Number.

The Logarithmical-Curve may alſo be conceived to be deſcribed by two Motions, one of which is equable, and the other accelerated or retarded according to a given Ratio.

For Example, if the Right Line A B, moves uniformly along the Line A N, ſo that the End A thereof deſcribes equal Spaces in equal Times; and, in the mean time, the ſaid Line A B ſo increaſes, that the Increments thereof generated in equal Times, be proportional to the whole increaſing Line, that is, if A B in going forward to $c\ d$, be encreaſed by the Increment $o\ d$, and in an equal Time, when it is come to C D, the Increment thereof is Dp, and D p to dc, is as do is to A B, that is, if the Increments generated in equal Times are always proportional to the Wholes; or, if the Line A B moving the contrary Way, diminiſhes in a conſtant Ratio, ſo that while it goes thro' the equal Spaces, Α Γ, Γ Π the Decrements A B — Γ Δ, Γ Δ — Π Ξ, are Proportionals to A B, Γ Δ. Then the End of the Line increaſing or decreaſing in the ſaid Manner, deſcribes the Logarithmical-Curve: For ſince A B: $d\,o$: : $d\,c$: D p : : D C: $f\,q$, it ſhall be (by Compoſition of Ratio) as A B: $d\,c$: : $d\,c$: D C : : D C : $f\,e$. and ſo on.

By theſe two Motions, *viz.* the one equable, and the other proportionally accelerated or retarded, the Lord *Neper* laid down the Origin of Logarithms, and call'd the Logarithm of the Sine of any Arc, *That Number which neareſt defines a Line that equally encreaſes, while, in the mean time, the Line expreſſing the whole Sine proportionally decreaſes to that Sine.*

It is manifeſt from this Deſcription of the Logarithmic-Curve, that all Numbers at equal Diſtances are continually proportional. It is alſo plain, that if there be four Numbers A B, C D, I K, L M, ſuch, that the Diſtance between the firſt and ſecond, be equal to the Diſtance between the third and the fourth: Let the Diſtance from the ſecond to the third be what it will, theſe Numbers will be proportional. For becauſe the Diſtances A C, I L, are equal, A B ſhall be to the Increment D *s*, as I K is to the Increment M T. Wherefore (by Compoſition) A B : D C : : I K: M L. And contrariwiſe, if four Numbers be proportional, the Diſtance between the firſt and the ſecond, ſhall be equal to the Diſtance between the third and the fourth.

The Diſtance between any two Numbers, is called the Logarithm of the Ratio of thoſe Numbers, and indeed doth not meaſure the Ratio itſelf, but the Number of Terms in a given Series of Geometrical Proportionals, proceeding from one Number to another, and defines the Number of equal Ratio's, by the Compoſition whereof the Ratio of Numbers are known.

If the Diſtance between any two Numbers be double to the Diſtance between two other Numbers, then the Ratio of the two former Numbers ſhall be the Duplicate of the Ratio of the two latter. For let the Diſtance I L between the Numbers I K, L M, be double to the Diſtance A *c*, between the Numbers A B, *c d*; and ſince I L is biſected in *l*, we have A *c* = I *l* = *l* L; and the Ratio of I K to *l m*, is equal to the Ratio of A B to *c d*; and ſo the Ratio of I K to L M, the Duplicate of the Ratio of I K to *l m*, (by *Def.* 10. *El.* 5.) ſhall be the Duplicate of the Ratio of A B to *c d*.

In like Manner, if the Diſtance E L be triple of the Diſtance A C, then will the Ratio of E F to L M, be triplicate of the Ratio of A B to C D: For becauſe the Diſtance is triple, there ſhall be three times more Proportionals from E F to L M, than there are Terms, of the ſame Ratio, from A B to C D; and the Ratio of E F to L M, as alſo of A B to C D, is compounded of the equal intermediate Ratio's, (by *Def.* 5. *El.* 6.) And ſo the Ratio of E F to L M, compounded of three times a greater Number of Ratio's, ſhall be triplicate of the Ratio of A B to C D. So likewiſe if the Diſtance G L be quadruple of the Diſtance A *c*, then ſhall the Ratio of G H to L M, be quadruplicate of the Ratio of A B to *c d*.

The Logarithm of any Number, is the Logarithm of the Ratio of Unity to that Number, or it is the Diſtance between Unity and that Number. And ſo Logarithms expreſs the Power, Place, or Order which every Number, in a Series of Geometrical-Progreſſionals obtains from Unity. For Example, if there be 10.000.000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10.000.000th Place from Unity; then it will be found, by Computation, that in the ſame Series from Unity, to 2 there are 3.010.300 proportional Terms, that is, the Number 2 will ſtand in the 3.010.300th Place. In like Manner, from Unity to 3, there will be found 4.771.213 proportional Terms, which Number defines the Place of the Number 3. The Numbers

10.000.000, 3.010.300, 4.771.213, ſhall be the Logarithms of the Numbers 10, 2, and 3.

If the firſt Term of the Series from Unity be called y, the ſecond Term will be y^2, the third y^3. *&c.* And ſince the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000} = 10$. Alſo $y^{3010300} = 2$. Alſo $y^{4771213} = 3$; and ſo on.

Wherefore all Numbers ſhall be ſome Powers of that Number which is the firſt from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Diſtances of Numbers from Unity, as has been ſhewn. The Logarithm of Unity ſhall be 0, for Unity is not diſtant from itſelf, but the Logarithms of Fractions are negative, or deſcending below nothing, for they go on the contrary Way. And ſo if Numbers increaſing proportionally from Unity, have poſitive Logarithms, or ſuch as are affected with the Sign $+$; then Fractions or Numbers in like Manner decreaſing, will have negative Logarithms, or ſuch as are affected with the Sign $-$; which is true when Logarithms are conſider'd, as the Diſtances of Numbers from Unity.

But if Logarithms take their Beginning not from an integral Unit, but from a Unit, that is in ſome Place of Decimal Fractions. For Example, from the Fraction $\frac{1}{10.000.000.000.000}$; then all Fractions greater than this, will have poſitive Logarithms, and thoſe that are leſs, will have negative Logarithms. But more ſhall be ſaid of this hereafter.

Since in the Numbers continually proportional, D C, E F, G H, I K, *&c.* the Diſtances C E, E G, G I, *&c.* are equal, the Logarithms A C, A E, A G, A I, *&c.* of thoſe Numbers ſhall be equidifferent, or the Differences of them ſhall be equal: And ſo the Logarithms of proportional Numbers are all in an Arithmetical-Progreſſion; and from hence proceeds that common Definition of Logarithms, that Logarithms are Numbers, which being adjoined to Proportions, have equal Differences.

In the firſt Kind of Logarithms that *Neper* publiſhed, the firſt Term of the continual Proportionals was placed only ſo far diſtant from Unity, as that Term exceeded Unity. For Example, if *v n* be the firſt Term of the Series from Unity A B, the Logarithm thereof, or the Diſtance A *n*, or B *y*, was, according to Him, equal to *v y*, or the Increment of the Number above Unity. As ſuppoſe *v y* be 1,000.0001, he placed 0,000.000.1 for it's Logarithm A *n*; and from hence, by Computation, the Number 10 ſhall be the 23.025.850th Term of the Series, which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expreſſes it's Diſtance from Unity in ſuch Parts whereof *v y* or A *n* is one.

But this Poſition is entirely at Pleaſure, for the Diſtance of the firſt Term may have any given Ratio to the Exceſs thereof above Unity, and according to that various Ratio (which may be ſuppoſed at Pleaſure,) that is between *v y* and B *y*, the Increment of the firſt Term above Unity, and the Diſtance of the ſame from Unity, there will be produced different Forms of Logarithms.

This firſt Kind of Logarithms was afterwards changed by *Neper*, into another more convenient one, wherein he put the Number 10, not as the 23.025.850th Term of the Series, but the 10.000.000th; and in this Form of Logarithms, the firſt Increment *v y* ſhall be to the Diſtance B *y*, or A *n*, as Unity, or AB, is to the Decimal-Fraction 0,4.342.994, which therefore expreſſes the Length of the Subtangent A T. *Fig.* 4.

After *Neper*'s Death, the excellent Mr. *Henry Briggs*, by great Pains, made and publiſhed Tables of Logarithms according to this Form. Now ſince in theſe Tables, the Logarithm of 10, or the Diſtance thereof from Unity, is 10.000.000, and 1,10,100,1000, 10.000, *&c.* are continual Proportionals, they ſhall be equidiſtant. Wherefore the Logarithm of the Number 100 ſhall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10.000 ſhall be 4,0000000; and ſo on.

Hence the Logarithms of all Numbers between 1 and 10, muſt begin with 0, or 0 muſt ſtand in the firſt Place to the Left-Hand; for they are leſs than the Logarithm of the Number 10, whoſe Beginning is Unity; and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and leſs than 2,0000000. Alſo the Logarithms between 100 and 1000, begin with 2, for they are greater than the Logarithm of 100, which begins with 2, and leſs than the Logarithm of a 1000 that begins with 3. In the ſame Manner it is demonſtrated, that the firſt Figure to the Left-Hand of the Logarithms between 1000 and 10.000, muſt be 3; and the firſt Figure to the Left-Hand of the Logarithms between 10.000 and 100.000, will be 4; and ſo on.

The firſt Figure of every Logarithm to the Left-Hand, is called the Characteriſtic or Index, becauſe it ſhews the higheſt or moſt remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the higheſt or moſt remote Place from Unity of the correſpondent Number to the Left-Hand, will be the Place of Tens. If the Index be 2, the moſt remote Figure of the correſpondent Number ſhall be in the ſecond Place from Unity, that is, it ſhall be in the Place of Hundreds; and if the Index of a Logarithm be 3, the laſt Figure of the Number anſwering to it, ſhall be in the Place of Thouſands. The Logarithms of all Numbers that are in Decuple or Subdecuple Progreſſion, only differ in their Characteriſtics, or Indices, they being written in all other Places with the ſame Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17.000, are the ſame, unleſs in their Indices; for ſince 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17.000; therefore the Diſtances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17.000, ſhall be all equal. And ſo ſince the Diſtance between 1 and 17, or the Logarithm of the Number

17 is 1.2304489, the Logarithm of the Number 170, will be = 2.2304489, and the Logarithm of the Number 1700 ſhall be 3.2304489, becauſe the Logarithm of the Number 100 = 2.0000000. In like Manner, ſince the Logarithm of the Number 1000 = 3.0000000, the Logarithm of the Number 17.000 ſhall 4.2304489.

So alſo the Numbers, 6748. 674, 8. 67, 48, 6, 748.0, 6748. 0, 06748, are continual Proportionals in the Ratio of 10 to 1; and ſo their Diſtances from each other ſhall be equal to the Diſtance or Logarithm of the Number 10, or equal to 1,0000000. And ſo ſince the Logarithm of the Number 6748 is 3,8291751, the Logarithms of the other Numbers ſhall be as in the Margin; where you may obſerve that the Indices of the laſt two Logarithms are only negative, and the other Figures poſitive; and ſo when thoſe other Figures are to be added, the Indices muſt be ſubſtracted, and contrariwiſe.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	—1,8292751
0,06748	—2,8291751

CHAP. II.

CHAP. II.

Of the Arithmetick of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BECAUSE, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product, the Diſtance between Unity and the Multiplier, ſhall be equal to the Diſtance between the Multiplicand and the Product; if therefore, the Number G H be to be multiplied by the Number E F, the Diſtance between G H and the Product, muſt be equal to the Diſtance A E, or to the Logarithm of the Multiplier; and ſo if G L be taken equal to A E, the Number L M ſhall be the Product, that is, if to the Logarithm of the Multiplicand A G be added, the Logarithm of the Multiplier A E, the Sum ſhall be the Logarithm of the Product.

In Diviſion, Unity is to the Diviſor, as the Quotient is to the Dividend, and ſo the Diſtance between the Diviſor and Unity, ſhall be equal to the Diſtance between the Dividend and the Quotient. So if L M be to be divided by E F, the Diſtance E A ſhall be equal to the Diſtance between L M and the Quotient, and ſo if L G be taken equal to E A, the Quotient will be at G; that is, if from A L, the Logarithm of the Dividend, be taken G L, or A E, the Logarithm of the Diviſor, there will remain A G, the Logarithm of the Quotient.

And from hence it appears, that whatſoever Operations in common Arithmetick are performed by multiplying or dividing of great Numbers, may be much eaſier, and more expeditiouſly done by the Addition or Subtraction of Logarithms.

Let, for Example, the Number 7589 be to be multiplied by 6757. Now, if the Logarithms of thoſe Numbers be added together, as in the Margin their Sum will be the Logarithm of the Product, whoſe Index 7 ſhews that there are ſeven Places of Figures, beſides Unity, in the Product; and in ſeeking this Logarithm in Tables, or the neareſt equal to it, I find that the Number anſwering thereto, which is leſs than the Product is 51278000, and the Number greater than the Product is 51279000, and if the adjoined Differences and proportional Parts be taken, the Numbers that muſt be added to the Place of Hundreds and Tens in the Product are 87, and that which muſt be added in the Place of Unity, will neceſſarily be 3, ſince ſeven times nine $=63$, and ſo the true Product ſhall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers

Log. 3. 8801846
Log. 3. 8297539.
Log. 7. 7099385.

Numbers to be added in the Place of Hundredths or Tenths, could not be had from thoſe Tables of Logarithms, which conſiſt of but 7 Places of Figures, beſides the Characteriſtick, and ſo in this Caſe, the *Vlacquian* or *Briggian* Tables ſhould be uſed; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

If the Number 78956 be to be divided by 278, by ſubtracting the Logarithm of the Diviſor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 anſwers; which therefore ſhall be the Quotient.

Log. 4. 8954004
Log. 2. 4440448
Log. 2. 4513556

Becauſe Unity, any aſſumed Number, the Square thereof, the Cube, the Biquadrate, *&c.* are all continual Proportionals, their Diſtances from each other ſhall be equal to one another. And ſo it is manifeſt, the Diſtance of the Square from Unity, is double of the Diſtance of its Root from the ſame: Alſo the Diſtance of the Cube, is triple of the Diſtance of its Root; and the Diſtance of the Biquadrate, is quadruple of the Diſtance of its Root from Unity, *&c.* And ſo if the Logarithm of any Number be doubled, we ſhall have the Logarithm of its Square; if it be tripled, we ſhall have the Logarithm of its Cube, and if it be quadrupled, the Logarithm of it's Biquadrate. And contrariwiſe, if the Logarithm of any Number be biſected, we ſhall have the Logarithm of the Square-Root thereof: Moreover, a third Part of the ſaid Logarithm, will be the Logarithm of the Cube-Root of the Number; and a fourth Part, the Logarithm of the Biquadrate-Root of that Number.

Hence, the Extraction of all Roots are eaſily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square-Root of 5, the half of 0,6989700 muſt be taken, and then that half 0.3494850 will be the Logarithm of the Square-Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2.23606 nearly anſwers.

CHAP. III.

Of the Arithmetick of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is neceſſary, for avoiding the Trouble of adding one Part of a Logarithm, and ſubtracting the other, that Logarithms do not begin from an integral Unit, but from ſome Unit that is the tenth or hundredth Place of Decimal-Fractions: For Example, let P O be $\frac{1}{10000000000}$ and from this let the Logarithms begin. Now this Fraction is ten times more diſtant from Unity to the Left-Hand, than the Number 10 is diſtant there from to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to P O. And ſo if A B be Unity, the Logarithm thereof, according to this Suppoſition, will not be 0, but O A will be = 10.0000000; for the Diſtance of any Tenth from Unity is 1.0000000, whence the Diſtance of the Number 10 from PO will be 11.0000000. Alſo the Diſtance of the Number 100 from P O, or it's Logarithm, beginning from P O, ſhall be 12.0000000, and the Logarithm of 1000, or the Diſtance from P O, will be 13.0000000. And thus, the Indices of all Logarithms are augmented by the Number 10; and thoſe Fractions whoſe Indices are — 1, or, — 2, or — 3, *&c.* are now made 9, 8, or 7, *&c.*

But if Logarithms begin from the Place of a Fraction, whoſe Numerator is Unity, and Denominator Unity with 100 Cyphers added to it, (which they muſt do when Fractions occur that are leſs than P O) then that Fraction will be 100 times more diſtant from Unity, than 10 is diſtant from it; and ſo the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and ſo on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than PO (whereat they begin) will be poſitive. And ſince the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{0000}$, &c. are in a continued Geometrical Progreſſion, they will be equally diſtant from each other; and accordingly their Logarithms will be equidifferent: And ſo when the Logarithm of 10, is 11.0000000, and the Logarithm of Unity is 10.0000000, and the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000, and in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Alſo for the ſame Reaſon, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be *99*, and the Index of the Logarithm of $\frac{1}{100}$ will be *98*, and the Index of Logarithm of the Fraction $\frac{1}{1000}$ ſhall be *97*, &c. And theſe Indices ſhew in what Place from Unity, the firſt Figure of the Fraction, not being a Cypher, muſt be put. For Example, if the Index be 4, the Diſtance thereof from the Index of Unity, (which is 10) *viz. 6*, ſhews that the firſt ſignificative Figure of the Decimal, is in the ſixth Place from Unity; and therefore, five Cyphers are to be prefixed thereto towards the Left-Hand. So alſo if the Index of Unity be 100, and the Index of the Fraction be 80, the firſt Figure thereof ſhall be in the 20th Place from Unity, and *19* Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction GH by the Fraction DC. Becauſe Unity is to the Multiplier, as the Mul-

tiplicand is to the Product; the Diſtance between Unity and the Multiplier ſhall be equal to the Diſtance between the Multiplicand and the Product. Therefore, if there be taken $GI = AC$, the Product IK ſhall be at I. And accordingly, if from OG, the Logarithm of the Multiplicand, there be taken GI or AC, there will remain OI, the Logarithm of the Product. But $AC = OA - OC$, which taken from OG, there will remain $OG + OC - OA = OI$, that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity, (which is always expreſſed by 10 or 100 with Cyphers) the Logarithm of the Product will be had. For Example, let the Decimal-Fraction 0,00734 be to be multiplied by the Fraction 0,000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margin, which being added together, and
the Logarithm of Unity being
taken away from the Sum, the 97,8656961
Remainder is the Logarithm 96,9425041
of the Product, whoſe Index 94 94,8082002
ſhews that the firſt Figure of
the Product is in the ſixth Place from Unity, and ſo there muſt be five Cyphers prefixed, and then the Product will be .00000642984

In Diviſion the Diviſor is to Unity, as the Dividend is to the Quotient; and ſo the Diſtance between the Diviſor and Unity ſhall be equal to the Diſtance between the Dividend and the Quotient. And ſo if the Fraction I K be to be divided by D C, you muſt take I G = C A, and the Place of the Quotient ſhall be G. But C A = O A — O C, which being added to O I, we have O A + O I — O C = O G, that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Diviſor, there will remain the Logarithm of the Quotient; ſo if the Number C D be to be divided by I K, you muſt take the Diſtance C S = I A, and then S T will be the Quotient, whoſe Logarithm is O A + O C — O I. Let C D = 0. 347, I K = 0. 00478. Then add the Logarithm of Unity to the Logarithm of C D, that is, put 1 or 10 before the Index thereof, and from that ſubtract the Logarithm of the Diviſor, and the Remainder will be the Logarithm of the Quotient, whoſe Index 11. ſhews that the Quotient is between the Numbers 10 and 100; and I ſeek the Number anſwering the Logarithm, which I find to be 72,549. If the Logarithm of a Vulgar-Fraction, for Example, $\frac{7}{8}$ be required, the Logarithm of Unity muſt be added to the Logarithm of the Numerator 7, or which is all one, you muſt put 10 or 100 before the Index thereof, and ſubduct from it the Logarithm of the Denominator 8, and there will remain the Logarithm of the Vulgar-Fraction $\frac{7}{8}$, or the Decimal .875

$$\begin{array}{r} 19.\ 5403295 \\ \underline{7.\ 6794279} \\ 11.\ 8609016 \end{array}$$

$$\begin{array}{r} 10.\ 8450980 \\ \underline{0.\ 9030900} \\ 9.\ 9420080 \end{array}$$

If the Powers of any Fraction DC be required, you muſt aſſume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually Proportional from Unity. Beſides, $AE = 2AC = 2AO - 2OC$, whence $OE = OA - AE = 2OC - AO$, that is, the Logarithm of the Square is the Double of the Logarithm of the Root, leſs the Logarithm of Unity. In like Manner, ſince $AG = 3AC = 3OA - 3OC$, we ſhall have $OG = OA - AG = 3OC - 2OA =$ the Logarithm of the Cube $=$ triple the Logarithm of the Root leſs the Double of the Logarithm of Unity. For the ſame Reaſon, becauſe $AI = 4AC = 4OA - 4OC$, we have $OI = 4OC - 3OA$, which is the Logarithm of the Biquadrate. And univerſally, if the Power of a Fraction be n, and the Logarithm L, then ſhall the Logarithm of the Power $n = nL - nOA + OA$, that is, if the Logarithm of a Fraction be multiplied by n, and from the Product be taken the Logarithm of Unity, multiplied by $n - 1$, the Logarithm of the Power n of that Fraction will be had.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{20} = .05$ the Logarithm of this Fraction is 8.6989700, which being multiplied by 6, gives the Number 52.1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, *viz.*

2. 1938200, to which the Number, 000000015625 anſwers. For the Index 2 ſhews that 7 Cyphers muſt be put before the firſt Figure.

If the 8th Power of the Fraction, .05 be required, by multiplying the Logarithm by 8, there will be produced *69*. 5917600, and ſince 70, which is ſeven times the Index of the Logarithm of Unity, cannot be taken from *69*, unleſs we run into negative Numbers, the Index of the Logarithm of Unity muſt be ſuppoſed 100, and then the Index of the Logarithm of the Fraction will be *98*. Now this Logarithm drawn into 8 gives 78*9*. 5917600, and if 700, which is 7 times the Index of the Logarithm of Unity be taken from 78*9*, there will remain 8*9*. 5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whoſe correſpondent Number is, .0000000000 39062, for ſince the Index is 8*9*, and the Difference thereof from 100 is 11; the firſt ſignificant Figure of the Fraction ſhall be in the 11th Place from Unity; and ſo there muſt 10 Cyphers be placed before it.

If the Roots of the Powers of Fractions be deſired, for Example, the Square Root of the Fraction E F, becauſe the Root is a mean Proportional between the Fraction and Unity, you muſt biſect A E in C, and then C D will be the Square Root of the Fraction E F. But $AC = \frac{1}{2} AE = \frac{OA - OE}{2}$, and ſo OC the Logarithm of the Root $= OA - AC = \frac{OA + OE}{2}$ And if the Cube Root of the Fraction G H be ſought, this ſhall be the firſt of two mean Proportionals between Unity and G H; and ſo if A G be divided into three equal Parts, the firſt of which is A C; then C D ſhall be the Root ſought, and becauſe $AC = \frac{1}{3} AG = \frac{OA - OG}{3}$, if this be taken from O A, there will remain $\frac{2OA + OG}{3} = OC =$ Logarithm of the Cube Root of the Fraction G H. So likewiſe the Biquadrat Root of the Fraction I K will be had, by dividing A I into four equal Parts, for the Root is the firſt of three mean Proportionals between Unity and the Fraction, and conſequently if $AC = \frac{1}{4} AI$, then will C D, be the Biquadrat Root of the Fraction I K. But $\frac{1}{4} AI = \frac{OA - OI}{4}$ and ſo

$$OC = OA - AC = \frac{3OA + OI}{4}$$

And univerſally, if the Root of any Power n of the Fraction L M be required, the Logarithm of the Root thereof will be $\frac{nOA - OA + OL}{n}$, that is, if the Number $n - 1$ be prefix'd to the Index of the Logarithm, and the Logarithm thus augmented be divided by n, the Quotient will give the Logarithm of the Root ſought. So if the Cube Root of the Fraction $\frac{1}{2}$ or .5 be ſought, you muſt place $2 = n - 1$ (ſince the Cube

Root is required) before the Logarithm thereof, and there will be had 29,6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$, and the Number ,7937 anſwering to this Logarithm, is the Root ſought.

CHAP. IV.

Of the Golden Rule of Proportion by Logarithms.

THE Rule of Proportion ſhews how, by having three Numbers given, a fourth Proportional to them may be found, *viz.* if the ſecond and third Terms be multiplied by one another, and the Product divided by the firſt Term, then will the Quotient be the fourth Proportional Term ſought. But this fourth Term is much eaſier found by Logarithms; for if the Logarithm of the firſt Term be taken from the Sum of the Logarithms of the ſecond and third Term, the Number remaining will be the Logarithm of the fourth ſought.

Or this may be done ſomething eaſier yet, if inſtead of the Logarithm of the firſt Term be taken it's Complement Arithmetical, or the Difference of that Logarithm, and the

Number 10. 0000000, which is done by ſetting down the Difference between each Figure of the Logarithm, and the Figure 9: for then if that Arithmetical-Complement be added to the Sum of the other two Logarithms, and if Unity, which is the firſt Figure to the Left-Hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term ſought; and ſo by this way Logarithms of the fourth Term is found by only one Addition of three Numbers. The Reaſon of this will be manifeſt from hence: Let there be three Numbers A, B, C, from which the firſt is to be taken from the Sum of the ſecond and third. Now this may not only be done by the common Way, but likewiſe, if there be any other third Number E, taken, and from this there be taken A, there will remain E — A, and if the Numbers B, C, and E — A be all added together, and from their Sum be taken E, there will remain B + C — A. So if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100, which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

85
23
108

Here follow ſome Trigonometrical-Examples of the Rule of Proportion ſolv'd by Logarithms.

Let A B C be a Right-lined Triangle, wherein are given, the Angle A *36* Degrees *46′*, the Angle B *98* Degrees *32′*, and the Side B C 3478, the Side A C is required. Say (by *Caſe* 1. of *Plain Trig.*) as the Sine of the Angle

Angle A is		
to the Sine	Arith. Comp. S, A.	0.2228938
of the An-	Log. Sin. B.	9.9951656
gle B, ſo is	Log. B C.	3.5413296
BC to AC.	Log. A C.	1.3.7593890

And becauſe the Logarithm Sine of the Angle A is the firſt Term of the Analogy, I ſubſtitute it's Complement-Arithmetical for the ſame, and add the Logarithm of B C, the Logarithm of S, B, and the ſaid Complement all three together, and reject Unity, which is in the firſt Place to the Left-Hand, and then the Logarithm of the Side A C will be given, and the Number anſwering thereto is 5706, 306 equal to the Side ſought A C.

Let there be a ſpherical Triangle A B C, in which are given all the Sides, *viz.* B C. = 30 Degrees, A B = 24 Degrees 4′, and A C = 42 Degrees 8′, the Angle B is required. Let B A be produced to M, ſo that B M = B C, then will A M the Difference of the Sides B C, B A, be equal to 5 Degrees 56′. Now (by *Caſe* 11. in Oblique-angled Spherical-Triangles) ſay, as the Rectangle under the Sines of the Legs, AB. BC. is to the Square of Radius, ſo is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$ to the Square of the Sine of one half the Angle B. But $\frac{AC+AM}{2}$ = 24 Degrees 2′, and $\frac{AC-AM}{2}$ = 18 Degrees 6′; and becauſe the firſt Term of the Analogy is the Rectangle under the Sine of A B, B C, and ſecond Term is the Square of Radius, the Sum of the Logarithm Sines of A B, B C, muſt be taken from double the Logarithm of Radius, and what remains muſt be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, & $\frac{AC-AM}{2}$, which is the ſame as if the Logarithm Sines of each of the Arcs A B, B C, were ſubtracted from the Log. of Radius, or if the Complements Arithmetical of theſe Sines be taken, & the Complements and the ſaid Sines be all added together; then ſhall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And ſo the half of the Logarithm 9.8965274 is the Logarithm Sine of half the Angle B = 51 Degrees 59′. 56″, and the Double of this Angle ſhall be 103 Degrees 59′, 52″ = B, which was ſought.

Log. S, B C Comp. Arith.	0.3010299
Log. S, A B Comp. Arith.	0.3898364
Log. S, $\frac{AC+AM}{2}$	9.6098803
Log. S, $\frac{AC-AM}{2}$	9.4923083
2. Log. S, Angle B.	19.7930549

CHAP. V.

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms any Term in a Series of Proportionals, either increaſing or decreaſing. Fig. 3.

IF any where in the Axis of the Logarithmical Curve, there be taken any Number of equal Parts S V, V Y, Y Q, *&c.* and at the Points S, V, Y, Q, *&c.* be raiſed the Perpendiculars S T, V X, Y Z, $Q\pi$, *&c.* then from the Nature of the Curve ſhall all theſe Perpendiculars be continually proportional; and therefore alſo the continual Increments Xx, Zz, $\pi\varpi$, ſhall be proportional to their Wholes: For ſince $ST : VX :: VX : YZ :: YZ : Q\pi$, it ſhall be (by Diviſion of Proportion) $ST : Xx :: VX : Zz :: YZ : \pi\varpi$, and (by Compoſition of Proportion) $VX : Xx :: YZ : Zz :: Q\pi : \pi\varpi$. Hence, if Xx be any part of any Right Line S T, then will Zz be the ſame Part of the Right

Line VX, and alſo $\pi\pi$ the ſame Part of the Right-Line YZ. For Example; if Xx be the $\frac{1}{20}$ Part of ST, then will $Zz = \frac{1}{20}\,VX$, and $\pi\pi = \frac{1}{20}\,YZ$; or which comes to the ſame, we ſhall have $VX = ST + \frac{1}{20}\,ST$, $YZ = VX + \frac{1}{20}\,VX$. Alſo $Q\pi = YZ + \frac{1}{20}\,YZ$.

Now make, as ST is to VX, ſo is Unity AB to NR; then ſhall $AN = SV$; and ſo each of the Right Lines SV, VY, YQ, &c. ſhall be equal to the Logarithm of RN, and AV, the Logarithm of the Term VX ſhall be equal to $AS + AN =$ Logarithm of $ST +$ Logarithm of NR. Alſo AY, the Logarithm of the Term YZ, ſhall be equal to $AS + 2\,AN =$ Logarithm $ST + 2$ Logarithm NR, and AQ, the Logarithm of the Term $Q\pi$ ſhall be equal to $AS + 3\,AN =$ Logarithm $ST + 3$ Logarithm NR. And univerſally, if the Logarithm of the Number NR be multiplied by a Number, expreſſing the Diſtance of any Term from the firſt, and the Product be added to the Logarithm of the firſt Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreaſing, that is, if the Terms diminiſh in a continual Ratio, and $Q\pi$ be the firſt Term; then the Logarithm of any other will be had, in multiplying the Logarithm of the Number NR, by a Number that expreſſes the Diſtance of it's Term from the firſt, and ſubtracting the Product from the Logarithm of the firſt. And if the ſaid Product be greater than the Logarithm of the firſt Term, beginning from Unity, then the Logarithms muſt begin from a Unit in ſome Place of Decimal Fractions, as from O P, and then the Logarithm of the Number Q n will be O Q.

Now let L M repreſent any Money, or Sum of Money put out to Intereſt, ſo that the Intereſt thereof be accounted but at the End of every Year, and let K *k* be the Gain or Intereſt thereof at the End of the firſt Year, then will I K be the Sum of the Intereſt and Principal. And again, I K becoming the Principal at the End of the firſt Year, H *h* which is proportional to I K, or in a conſtant Ratio, will be the Gain at the end of the ſecond Year; and ſo H G, at the End of the ſecond Year, will become the Principal; and at the end of the third Year F *f*, proportional to G H, will be the Gain. Now let us ſuppoſe the Principal be augmented every Year $\frac{1}{20}$ Part thereof, ſo that $IK = LM + \frac{1}{20}LM$, $GH = IK + \frac{1}{20}IK$, $EF = GH + \frac{1}{20}GH$, and ſo on. And accordingly the Terms L M, I K, G H, E F, *&c.* continual Proportionals, it is required to find the Amount of the Money at the End of any Number of Years.

Let L M be a Farthing. Becauſe L M is to I K, as 1 to $1 + \frac{1}{20}$, or as 1 to 1.05, as A B is to N R, then will N R = 1.05, whoſe Logarithm A N. is 0.0211893, or more accurately 0.0211892991, it is required to find the Amount of a Farthing put out at compound Intereſt, at the End of 600 Years, multiply A N by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$, *viz.* 97.0177288 (for a Farthing is $\frac{1}{960}$ Part of a Pound) and the Sum 109. 7313082 ſhall be the Logarithm of the Number ſought; and ſince the Index 109 exceeds the Index of Unity by 9, there ſhall be nine Places of Figures above Unity in the correſpondent Number, and that Number being ſought in the Tables, will be found greater than 5386500000, and leſs than 5386600000. And therefore a Farthing put out at Intereſt upon Intereſt, at 5 *per Cent.* per *Annum*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up by all the Gold and Silver that has been dug out of the Bowels of the Earth, from the beginning of the World to this Time.

Let $Q\pi$ expound any Sum of Money due to ſome Perſon at the End of a full Year, but without Intereſt. Now it is certain, that if the Debtor ſhould pay down preſent the whole Sum of Money, he would loſe the yearly Uſury or Intereſt that his Money would gain him; and ſo a leſſer Sum, being put out to Intereſt, will at the End of one Year, together with the Intereſt thereof, be equal to the Sum of Money $Q\pi$. Now this preſent Sum of Money, which together with the Intereſt thereof is equal to the Sum of Money $Q\pi$, is called the preſent Worth of the Money $Q\pi$. Let A N be the Logarithm of the Ratio, which the Principal has to the Sum of the Principal and Intereſt, that is, if the Principal be twenty times the yearly Intereſt, let A N be the Logarithm of the Number $1 + \frac{1}{20}$ or 1. 05, and take Q I equal to A N; then will A Y be the Logarithm of the preſent Worth of Money Q π. For it is manifeſt, that the Money Y Z put out to Intereſt, will at the End of one Year amount to the Money Q π, and ſo to have the Logarithm of the preſent Worth thereof, or Y Z, the Logarithm A N, muſt be taken from the Logarithm A Q, and there will remain the Logarithm A Y of the preſent Worth, or Y Z. But if the Sum π Q be not due till the End of two Years, then the Logarithm 2 A N muſt be ſubtracted from the Logarithm A Q, and there will remain A V, the Logarithm of the preſent Worth, or of the Sum that muſt be paid down preſent for the Money Q π due at the End of two Years. For it is manifeſt, that the Money V X being put out to Intereſt, will at the End of two Years amount to the Sum of Money Q π. By the ſame Reaſon, if the Sum Q π, be not due until the End of three Years, the Logarithm 3 A N muſt be ſubtracted from the Logarithm of Q π, and the Remainder A S, ſhall be the Logarithm of the Number S T, or S T ſhall be the preſent Worth of the Sum Q π due at the three Years end. And univerſally, if the Logarithm A N be multiplied by the Number of Years, at the End of which the Sum Q π is due, and the Number produced be taken from the Logarithm A Q, then will the Logarithm of the preſent Worth of the Sum Q π be had. And from hence it is manifeſt, if 5386500000 Pounds be due to ſome Society at the End of 600 Years, then would the preſent Worth of that vaſt Sum of Money be ſcarcely a Farthing.

If the proportional Right Lines H G, E F, A B, C D, *Fig.* 4. are Ordinates to the Axis of the Logarithmical-Curve, and if their Ends F H, D B, be joined by Right Lines, which produced meet the Axis in the Points P and K, then the Right Lines G P, A K, will be always equal. For ſince G H : E F : : A B : C D. it will be as G H : F *s* : : A B : D R. But becauſe of the equiangular Triangles P G H, H *s* F, as alſo K A B, B R D, we have P G : H *s* : : (G H : F *s* : : A B : D R : :) K A : B R. And ſince the Conſequents H *s*, B R, are equal, the Antecedents P G, K A, ſhall be alſo equal. W. W. D.

If the Right Lines C D, E F, equally accede to A B, G H, ſo that the Point D at laſt may coincide with B, and the Point F with H, then the Right Lines D B K, F H P, which did cut the Curve before, will be changed into the Tangents B T, H V. And the Right Lines A T, G V, will be always equal to each other, that is, the Portion of the Axis A T, or G V, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every where be a conſtant and given Length. And this is one of the chief Properties of the Logarithmical Curve; for the different Species or Forms of thoſe Curves are determined by the Subtangents.

The Logarithms or the Diſtances from Unity of the ſame Number, in two Logarithmical-Curves of different Species, will be proportional to the Subtangents of their Curves. For let H B D, S N Y, *Fig.* 4, 5. be

Curves, whoſe Subtangents are A T, M X, and let A B = M N = Unity; alſo D C = Q Y, then ſhall A C the Logarithm of the Number C D, in the Logarithmical-Curve H D be to M Q, the Logarithm of the Number Q Y, (or of the ſaid C D,) in the Curve S Y, as the Subtangent A T is to the Subtangent M X. For let there be ſuppoſed an infinite Number of mean proportional Terms between A B, C D, or N M, Q Y, in the Ratio of A B to *a b*, or M N to *m n*; and ſince A B = M N, then will *a b* = *m n*, as alſo *b c* = *n o*. And becauſe the Number of proportional Terms in each Figure are equal, they do divide the Lines A C, M Q, into equal Numbers of Parts, the firſt of which A *a*, M *m*, and ſo the ſaid Parts ſhall be proportional to their Wholes, that is, it will be as A *a* : M *m* : : A C : M Q. And becauſe the Triangles T A B, B *c b*, are ſimilar, (for the Part of the Curve B *b* nearly coincides with the Portion of the Tangent,) as alſo the Triangles X M N, N *o n*, are ſimilar, we have A *a*, or B *c* : *b c* : : T A : A B.

Alſo as *no*, or *bc* : N*o* : : M N, or A B : M X.

Where (by Equality of Proportion) it will be B*c* : N*o* : : T A : M X : : A*a* : M*m* : : A C : M Q; *which was to be demonſtrated.* If A T be call'd *a*, ſince A B : A T : : *bc* : B*c*, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a ſmall matter exceeding it, be given, then will the Subtangent of the Logarithmical-Curve be had. For the Exceſs bc is to the Logarithm Bc, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference ſhall be to the Difference of their Logarithms, as one of the Numbers is to the Subtangent. For Example; if the Increment bc be ,00000 00000 00001 02255 31945 60259, and Bc or Aa the Logarithm of the Number ab be ,00000 00000 00000 44408 92098 50062. Now if a fourth Proportional be found to the ſaid two Numbers and Unity, *viz.* 43429 44819 03251, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve, expreſſing *Brigg's* Logarithms.

If a Sum of Money be put out to Intereſt on this Condition, That a proportional Part of the yearly Rate of Intereſt thereof, be accounted every Moment of Time, *viz.* ſo, that at the End of the firſt Moment of Time, or indefinitely ſmall Particle of a Year, the Intereſt gotten thereby, be proportional to that Time; which being added to the Principal, again begets Intereſt at the End of the ſecond Moment of Time, and then the Principal, and this Intereſt become a Principal, and ſo on. It is requir'd to find the Amount of that Sum at the Year's End. Let a be nearly the Intereſt of Unity, or of one Pound. Then if one whole Year, or 1 gives the Intereſt a, the indefinitely ſmall Particle of a Year Mm, will give the Intereſt $Mm \times a$, proportional to Mm; and accordingly, if Unity be expounded by MN, the firſt Increment thereof ſhall be $no=Mm\times a$. This being granted, let a Logarithmical-Curve be ſuppos'd to be deſcribed through the Points Nn, whoſe Axis is OMQ. Then in this Curve, if the Portion of the Axis MQ expreſſes the Time, the ordinate Qy will repreſent the Money proportionally increaſing every Moment, to that Time. For if there be taken ml, &c.$=Mm$, the Ordinates lp, &c. ſhall be in a Series of continual Proportionals in the Ratio of MN to mn, that is, they increaſe in the ſame Ratio, as the Money doth.

Again, let the Right-Line NX touch the Logarithmical-Curve in N, and the Subtangent thereof MX, ſhall be conſtant and invariable, and the ſmall Triangle Non, ſhall be ſimilar to the Triangle XMN. But it has been prov'd, that the Increment $no=Mm\times a=No\times a$; and ſo $no : No :: No\times a : No :: a : 1$. But as no is to No, ſo ſhall NM be to MX. Wherefore it ſhall be as a is to 1, ſo is NM, or 1, to $MX=\frac{1}{a}=$ Subtangent.

Now if the yearly Rate of Intereſt be $\frac{1}{20}$ part of the Principal, or if $a=\frac{1}{20}=.05$, then will $MX=\frac{1}{a}=20$.

Becauſe in different Forms of Logarithms, the Logarithms of the ſame Number, are proportional to the Subtangents of their Curves: If MQ expreſſes the Time of a whole Year, or Unity, then ſhall QY be the Amount of the Money at the Year's End. And to find QY, ſay, as MX, or 20 is to 0.4342944, (which Number expounds the

Subtangent of the Logarithmical-Curve, expreſſing *Brigg*'s Logarithms) ſo is one Year or Unity to a *Briggian* Logarithm, anſwering to the Number Q Y. This Logarithm will be found 0.0217147, and the Number anſwering to the ſame is 1.05127=Q Y, whoſe Increment above Unity, or the Principal, exceeds the yearly Intereſt ,05 but a ſmall matter. And ſo if the yearly Intereſt of 100 Pounds be 5 Pounds, the proportional yearly Intereſt, which is added to the Principal 100, at the end of each Particle of the Year, will amount only at the Year's End to 5 Pounds, 2 Shillings and $6\frac{1}{2}$ Pence.

And if ſuch a Rate of Intereſt be requir'd, that every Moment a Part of it continually proportional to the increaſing Principal, be added to the Principal, ſo that at the Year's End, an Increment be produc'd, that ſhall be any given Part of the Principal, for Example, the $\frac{1}{20}$ Part, ſay, as the Logarithm of the Number 1.05 is to 1; that is, as 0.0211893 is to 1, ſo is the Subtangent 0.4342944 to $\frac{1}{a} = 20.49$, and then will $a = \frac{1}{20,49} = .0488$. For if ſuch a Part of the Rate of Intereſt .0488 be ſuppoſed as anſwers to a Moment, that is, having the ſame Ratio to .0488, as a Moment has to a Year, and it be made, as Unity is to that Part of the Rate of Intereſt, ſo is the Principal to the Momentaneous-Increment thereof; then will the Money continually increaſing in that manner, be augmented at the Year's-End the $\frac{1}{20}$ Part thereof.

CHAP. VI.

Of the Method by which Mr. Brigg's *computed his Logarithms, and the Demonſtration thereof.*

ALTHO' Mr. *Briggs* has no where deſcrib'd the Logarithmical-Curve, yet it is very certain, that from the Uſe and Contemplation thereof, the Manner and Reaſon of his Calculations will appear. In any Logarithmical-Curve H B D, let there be three Ordinates A B, ab, qs, nearly equal to one another; that is, let their Differences have a very ſmall Ratio to the ſaid Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For ſince the Ordinates are nearly equal to one another, they will be very nigh to each other, and ſo the Part of the Curve B s, intercepted by them, will almoſt coincide with a ſtrait Line; for it is certain, that the Ordinates may be ſo near to each other, that the Difference between the Part of the Curve, and the Right-Line ſubtending it, may have to that Subtenſe, a Ratio leſs than any given Ratio. Therefore the Triangles B*cb*, B*rs*, may be taken for Right-lin'd, and will be Equiangular. Wherefore, as *sr* : *bc* : : B*r* : B*c* : : A*q* : A*a*; that is, the Exceſſes of the Ordinates, or Lines above the leaſt, ſhall be proportional to the Differences of their Logarithms. And from hence appears the Reaſon of the Correction of Numbers and Logarithms, by Differences and proportional Parts. But if AB be Unity, the Logarithms of Numbers ſhall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10, or which is the ſame thing, if the Square-Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof ſhall be $\frac{1}{2}$ of the Logarithm of 10, and ſo will be given. If again, between the Number before found, and Unity, there be found a mean Proportional, which may be done in extracting the Square-Root of the ſaid Number, this Number or Root will be twice nearer to Unity than the former, and its Logarithm will be one half of the Logarithm of that, or one fourth of the Logarithm of 10. And if in this Manner, the Square-Root be continually extracted, and the Logarithms biſected, you will at laſt get a Number, whoſe Diſtance from Unity, ſhall be leſs than the $\frac{1}{1\,00000\,00000\,00000}$ part of the Logarithm of 10. And after Mr. *Briggs* had made 54 Extractions of the Square-Root, he found the Number 1.00000

00000 00000 12781 91493 20032 3442, and its Logarithm was 0,00000 00000 00000,05551 11512312578270 2. Suppoſe this Logarithm to be equal to Aq or Br, and let qs be the Number found, by extracting the Square-Root; then will the Exceſs of this Number above Unity, *viz.* $rs =$,00000 00000 00000. 12781 91493 20032 34.

Now by means of theſe Numbers, the Logarithms of all other Numbers may be found in the following Manner: Between the given Number (whoſe Logarithm is to be found) and Unity, find ſo many mean Proportionals, (as above,) till at laſt a Number be gotten ſo little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of ſignificant Figures after thoſe. Let this Number be ab, and let the ſignificant Figures, with the Cyphers prefixed before them, denote the Difference bc. Then ſay, As the Difference rs, is to the Difference bc, ſo is Br a given Logarithm, to Bc, or Aa, the Logarithm of the Number ab; which therefore is given. And if this Logarithm be continually doubled, the ſame Number of Times, as there were Extractions of the Square-Root, you will at laſt have the Logarithm of the Number ſought. Alſo by this Way, may the Subtangent of the Logarithmical-Curve be found, *viz.* in ſaying, As $rs : Br :: AB$, or Unity : AT, the Subtangent, which therefore will be found to be 0.43429 44819 03251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards,

wards, as alſo its Logarithm, and the Logarithm of another Number ſufficiently near to N M be ſought, ſay, As N M is to the Subtangent X M, ſo is *n o* the Difference of the Numbers to N *o* the Difference of the Logarithms. Now, if N M be Unity = A B, the Logarithms will be had by multiplying the ſmall Differences *b c* by the conſtant Subtangent A T.

By this Way may be found the Logarithms of 2, 3, and 7, and by theſe the Logarithms of 4, 8, 16, 32, 64, *&c.* 9, 27, 81, 243, *&c.* as alſo 7, 49, 343, *&c.* And if from the Logarithm of 10, be taken the Logarithm of 2, there will remain the Logarithm of 5, ſo there will be given the Logarithms of 25, 125, 625, *&c.*

The Logarithms of Numbers compounded of the aforeſaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, *&c.* are eaſily had by adding together the Logarithms of the component Numbers.

But ſince it was very tedious and laborious, to find the Logarithms of the Prime-Numbers, and not eaſy to compute Logarithms by Interpolation, by firſt, ſecond and third, *&c.* Differences, therefore the great Men, Sir *Iſaac Newton*, *Mercator*, *Gregory*, *Wallis*, and laſtly, Dr. *Halley*, have publiſhed infinite converging Series, by which the Logarithms of Numbers to any Number of Places, may be had more expeditiouſly and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philoſophical Tranſactions*, wherein he has demonſtrated thoſe Series after a new Way, and ſhews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by means of which may be found eaſily and expeditiouſly, the Logarithms of large Numbers.

Let z be an odd Number, whoſe Logarithm is ſought; then ſhall the Numbers $z - 1$ and $z + 1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called y: Therefore, alſo the Logarithm of a Number, which is a Geometrical-Mean between $z - 1$ and $z + 1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series

$$y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9} \&c$$

ſhall be equal to the Logarithm of the Ratio, which the Geometrical-Mean between the Numbers $z - 1$ and $z + 1$, has to the Arithmetical-Mean, *viz.* to the Number z.

If the Number exceeds 1000, the firſt Term of the Series $\frac{y}{4z}$ is ſufficient for producing the Logarithm to 13 or 14 Places of Figures, and the ſecond Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the firſt Term will exhibit the Logarithm to 18 Places of Figures; and ſo this Series is of great Uſe in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the ſame as the Logarithm of 2, with the Index 4 prefix'd to it; and the

M S.

shews how
them. But
here to ad-
may be
Loga-

Loga-
numbers
accurately
Case of the
or called y-
Number,
[illegible]
ual to the
[illegible]
.

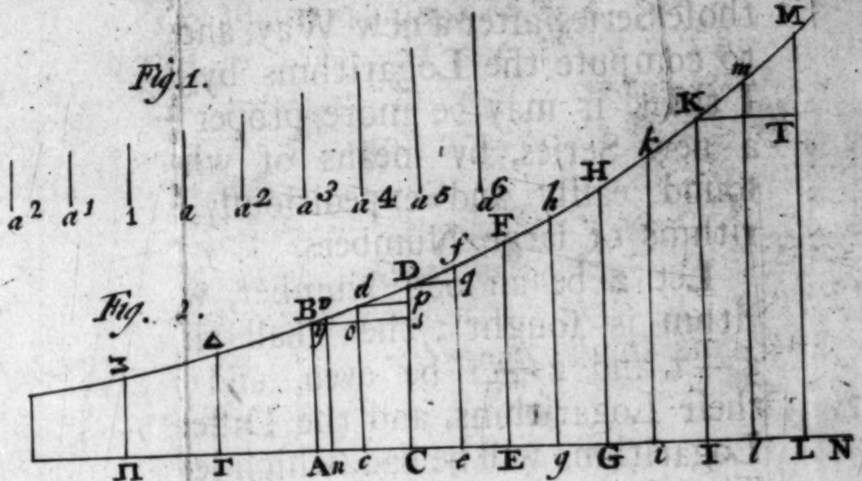
[illegible]
of the Ra-
in between
ing to the
other
ed Term

producing
of Figures
[illegible]
re great-
exhibit the
[illegible]
ing up the
[illegible] by
[illegible] to
the Loga-
Logarithm
to the end
the

The last Page

Fig. 1.

Fig. 2.



a^2 a^1 1 a a^2 a^3 a^4 a^5 a^6
M m K T k H h F f q D d p s B v y o
Ʃ Δ
Π Γ A n c C e E g G i I l L N

Chap. 4.

Fig. 3.

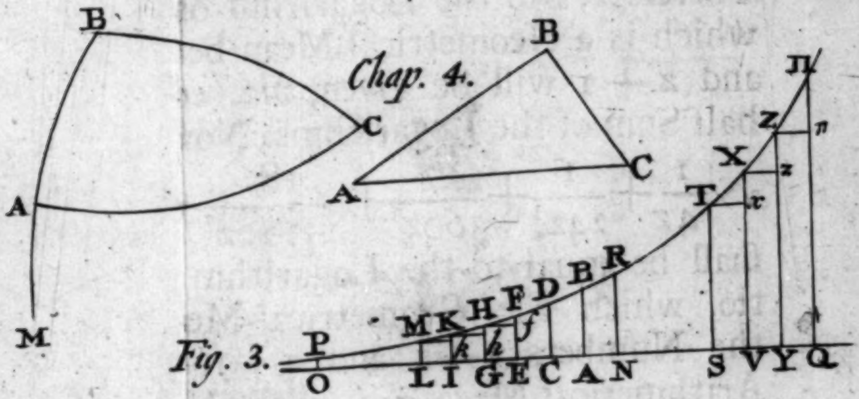


Fig. 4.

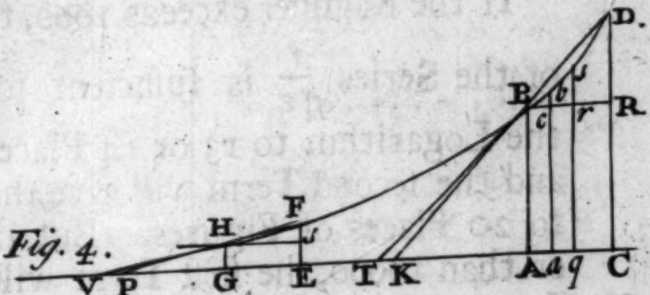
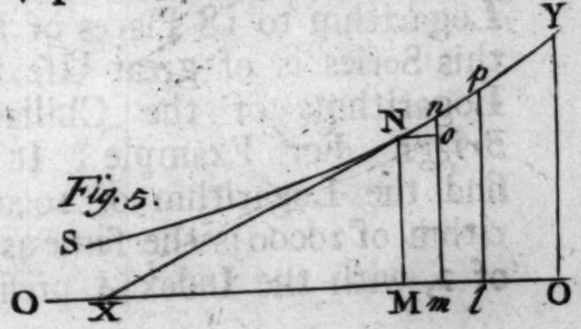


Fig. 5.



the Difference of the Logarithms of 20000 and 20002, is the ſame as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0. 00004 34272 7687. And if this Difference be divided by 4*z*, or 80004, the Quotient $\frac{y}{4z}$ ſhall be ---- 0. 00000 00005 42813

And if the Logarithm of the Geometrical-Mean be added to the Quotient, 4. 30105 17093 02416

4. 30105 17098 45230

Quotient, the Sum will be the Logarithm of 20001. Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond ſix Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Tables, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000, are to be found by this Way, the Labour of doing them, will moſtly conſiſt in ſetting down the Numbers. *Note*, This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a mind to be inform'd more in this Matter, let them conſult his abovenam'd Treatiſe.

FINIS.

An *Explanation* of the uſual *Algebraic* Signs or Characters.

+	*More*, or to be added to
−	*Leſs*, or to be ſubtracted from
=	*Equal* to
×	*Multiplied* by
$\frac{a}{b}$	a *divided* by b
a : b :: c : d	As a is to b, ſo is c to d
° ′ ″ ‴	*Degrees*, *Minutes*, *Seconds*, *Thirds*
$52^{2}.\ 44^{3}.\ 3^{4}$	52 *Seconds*, 44 *Thirds*, 3 *Fourths*
R.	*Radius*, or Sine of 90 Degrees
S.	*Sine*
Cos.	*Coſine*, or *Sine* of the *Complement*
T.	*Tangent*
Cot.	*Cotangent*, or *Tangent* of the *Complement.*
V. S.	*Verſed-Sine*
□ or q	*Square*
$\surd$ or $\surd^{2}$	*Square* Root
A^{3} or B^{4}	*Cube* of A. or 4*th Power* of B.
$\surd^{3}$ or $\surd^{4}$	*Cube* Root, or *Biquadrate* Root (*Euclid*
by 4. 6	by the 4*th* Propoſition of the 6*th* Book of
W. W. D.	*which was* to be *demonſtrated* or *done*
N°. S°.	*North* or *South*
♈ ♎	*Aries* or *Libra*
♋ ♑	*Cancer* or *Capricorn*
☉ ☽	*Sun* or *Moon.*

BRITISH MUSEUM 6 NO73